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THE TIME EVOLUTION OF LARGE SYSTEMS: INVESTIGATIONS OF A SOLUBLE MODEL

by

ERIC HIOB ©

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled THE TIME EVOLUTION OF LARGE SYSTEMS: INVESTIGATIONS OF A SOLUBLE MODEL submitted by Eric Hiob in partial fulfillment of the requirements for the degree of Master of Science in Theoretical Physics.

DEDICATION

To my mother and father

ABSTRACT

This thesis is concerned with the time evolution of a large system consisting of several kinds of particles that can be transformed into each other by an external field. Several kinds of systems are studied, and in each case an exact equation describing the time evolution of the system is obtained.

Chapter One is an introduction to the problem. It is a recapitulation of some work done by H.J. Kreuzer and K. Nakamura [1]. They considered two kinds of Bose-Einstein particles a and b, which can be transformed into each other via a constant external field. We extend this to include Fermi-Dirac particles.

Chapter Two generalizes the work of Chapter One for bosons to a three-channel model (i.e. with three kinds of particles a, b, and c). The resulting time evolution equation is then studied and illustrated by various examples. The extension to an n-channel system of arbitrary topology is easily made. A time evolution equation is derived but not studied.

The zero temperature limit for fermions is obtained in Chapter Three. Kelvin's stationary phase argument is used to study the asymptotic time evolution.

In Chapter Four we study a two-channel system initially in equilibrium but with the a particles at temperature T_a and the b particles at temperature T_b.

Finally in Chapter Five a high temperature expansion is made for weakly degenerate two-channel boson and fermion systems.

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I would like to express my gratitude to all those who helped me during my studies and the writing of this thesis; especially my wife, Lidia.

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TABLE OF CONTENTS

CHAPTER I

THE TWO CHANNEL SYSTEM

We consider two kinds of particles a and b (either both bosons or both fermions) which can be transformed into each other via an external field.

The hamiltonian

The hamiltonian for this sytem in coordinate representation is

$$\mathcal{H} = \frac{\hbar^2}{2m_1}\int \vec{\nabla}\psi_1^\dagger \cdot \vec{\nabla}\psi_1 d\vec{x} + \frac{\hbar^2}{2m_2}\int \vec{\nabla}\psi_2^\dagger \cdot \vec{\nabla}\psi_2 d\vec{x}$$

$$+ \theta(t)\int d\vec{x}\ V(\vec{x})\,[\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1] \qquad (1.1)$$

where $\psi_i = \psi_i(x,t)$, $i = 1,2$ are either both Bose or both Fermi field operators and

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} .$$

Fourier decomposing the field operators, we get

$$\psi_1(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} a_{\vec{k}}\, e^{i\,\vec{k}\cdot\vec{x}}$$

$$\psi_2(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} b_{\vec{k}}\, e^{i\,\vec{k}\cdot\vec{x}} \qquad (1.2)$$

<u>NOTE</u> Henceforth, the wavenumber $\vec{k}$ will simply be written k, the dot product k·x, etc. Also

$$\sum_k \equiv \sum_{(k_x,k_y,k_z)}$$

where

$$k_i = \frac{2\pi n_i}{V^{1/3}}\ , \quad i = x,y,z; \quad n_i = 0,1,\ldots\infty .$$

Substituting (1.2) into (1.1), we get the hamiltonian in creation- and annihilation-operator representation

$$\mathcal{H} = \sum_k \varepsilon_k^{(1)} a_k^\dagger a_k + \sum_k \varepsilon_k^{(2)} b_k^\dagger b_k + \theta(t) \sum_{k,k'} V_{kk'} (a_k^\dagger b_{k'} + b_k^\dagger a_{k'}) \tag{1.3}$$

where

$$\varepsilon_k^{(i)} = \frac{\hbar^2 k^2}{2m_i}$$

$$V_{kk'} = \frac{1}{V} \int dx\ V(x) e^{i(k-k')\cdot x} \quad .$$

Note that if $V(x) = V_o$ (a constant interaction) then $V_{kk'} = V_o \delta_{kk'}$.

<u>Diagonalization and exact solution</u>

In matrix notation, the hamiltonian equation is

$$\mathcal{H} = (a_k^\dagger | b_k^\dagger) \begin{pmatrix} E^{(1)} & V \\ V^\dagger & E^{(2)} \end{pmatrix} \begin{pmatrix} a_k \\ \hline b_k \end{pmatrix} \tag{1.4}$$

where $E^{(i)}$ and V are submatrices with elements $E^{(i)}_{kk'} = \varepsilon^{(i)}_{k} \delta_{kk'}$ and $V_{kk'}$ respectively; and $(a^\dagger_k | b^\dagger_k)$ represents the infinite vector $(\ldots a^\dagger_k \ldots | \ldots b^\dagger_k \ldots)$ with all wavenumbers $\vec{k}$ included. The hamiltonian equation (1.4) can be diagonalized with a unitary matrix

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} .$$

That is

$$\mathcal{H} = [(a^\dagger_k | b^\dagger_k) U^{-1}] \left(U \begin{pmatrix} E^{(1)} & V \\ V^\dagger & E^{(2)} \end{pmatrix} U^{-1} \right) \left(U \left(\frac{a_k}{b_k} \right) \right)$$

$$= (\alpha^\dagger_k | \beta^\dagger_k) \Lambda \left(\frac{\alpha_k}{\beta_k} \right) \tag{1.5}$$

where

$$U \begin{pmatrix} E^{(1)} & V \\ V^\dagger & E^{(2)} \end{pmatrix} U^{-1} = \begin{pmatrix} \Lambda^{(1)} & 0 \\ 0 & \Lambda^{(2)} \end{pmatrix} = \Lambda \tag{1.6}$$

and $\Lambda^{(i)}$ are diagonal submatrices with elements $\Lambda^{(i)}_{kk'} = \lambda^{(i)}_k \delta_{kk'}$. We introduce quasi-particles by a linear transformation of the annihilation operators

$$\begin{aligned} \alpha_k &= \sum_{k'} (A_{kk'} a_{k'} + B_{kk'} b_{k'}) \\ \beta_k &= \sum_{k'} (C_{kk'} a_{k'} + D_{kk'} b_{k'}) \end{aligned} \tag{1.7}$$

with inverse transformations

$$a_k = \sum_{k'} (A^*_{k'k}\alpha_{k'} + C^*_{k'k}\beta_{k'})$$

$$\tag{1.8}$$

$$b_k = \sum_{k'} (B^*_{k'k}\alpha_{k'} + D^*_{k'k}\beta_{k'})$$

and similar equations for the creation operators. If the particle operators satisfy the Bose (Fermi) commutation relations, then the quasi-particle operators again satisfy the Bose (Fermi) commutation relations. This is guaranteed by the unitarity of U.

<u>NOTE</u>: In the following theorems, the upper sign applies to Fermions, the lower sign to Bosons: $\frac{F-D}{B-E}$ $\pm$

<u>Theorem 1</u>

If I. $[a_k, a^\dagger_{k'}]_\pm = \delta_{kk'}$

II. $[b_k, b^\dagger_{k'}]_\pm = \delta_{kk'}$

III. $[a_k, b^\dagger_{k'}]_\pm = 0 \qquad \frac{F-D}{B-E}$ (1.9)

IV. $[a_k, a_{k'}]_\pm = 0$

V. $[b_k, b_{k'}]_\pm = 0$

all hold, then all of the following are true:

I. $[\alpha_k, \alpha^\dagger_{k'}]_\pm = \delta_{kk'}$

II. $[\beta_k, \beta^\dagger_{k'}]_\pm = \delta_{kk'}$

III. $[\alpha_k, \beta^\dagger_{k'}]_\pm = 0 \qquad \frac{F-D}{B-E}$ (1.10)

IV. $[\alpha_k, \alpha_{k'}]_\pm = 0$

V. $[\beta_k, \beta_{k'}]_\pm = 0$.

We shall prove only (1.10) I and V. Proofs for the rest are similar.

Proof:

I. $[\alpha_k, \alpha^\dagger_{k'}]$

$$= \sum_{qq'} \{ (A_{kq}a_q + B_{kq}b_q)(A^*_{k'q'}a^\dagger_{q'} + B^*_{k'q'}b^\dagger_{q'})$$

$$\pm (A^*_{k'q'}a^\dagger_{q'} + B^*_{k'q'}b^\dagger_{q'})(A_{kq}a_q + B_{kq}b_q) \} \qquad \frac{F-D}{B-E}$$

$$= \sum_{qq'} (A_{kq}A^*_{k'q'}\delta_{qq'} + B_{kq}B^*_{k'q'}\delta_{qq'})$$

$$+ \sum_{qq'} (A_{kq}B^*_{k'q'}0 + B_{kq}A^*_{k'q'}0)$$

$$= \sum_q (A_{kq}A^*_{k'q} + B_{kq}B^*_{k'q})$$

$$= \delta_{kk'}$$

V. $[\beta_k, \beta_{k'}]_\pm$

$$= \sum_{qq'} \{ (C_{kq}a_q + D_{kq}b_q)(C_{k'q'}a_{q'} + D_{k'q'}b_{q'})$$

$$\pm (C_{k'q'}a_{q'} + D_{k'q'}b_{q'})(C_{kq}a_q + D_{kq}b_q) \} \qquad \frac{F-D}{B-E}$$

$$= \sum_{qq'} \{ C_{kq}C_{k'q'}[a_q, a_{q'}]_\pm + D_{kq}D_{k'q'}[b_q, b_{q'}]_\pm$$

$$+ D_{kq}C_{k'q'}[b_q, a_{q'}]_\pm + C_{kq}D_{k'q'}[a_q, b_{q'}]_\pm \}$$

$$= 0$$

Q.E.D.

We shall also need the following results:

Theorem 2

If $\mathcal{H} = \sum_k \lambda_k^{(1)} \alpha_k^\dagger \alpha_k + \sum_k \lambda_k^{(2)} \beta_k^\dagger \beta_k$

then

$$\text{I.}\quad \alpha_{k'}\mathcal{H} = \mathcal{H}\alpha_{k'} + \lambda_{k'}^{(1)} \alpha_{k'}$$

(1.11)

$$\text{II.}\quad \beta_{k'}\mathcal{H} = \mathcal{H}\beta_{k'} + \lambda_{k'}^{(2)} \beta_{k'} \;.$$

Proof: We will prove I only. The proof for II is the same.

$$\alpha_{k'}\mathcal{H} = \sum_k \{\lambda_k^{(1)} (\alpha_{k'}\alpha_k^\dagger)\alpha_k \mp \lambda_k^{(2)} \beta_k^\dagger \alpha_{k'}\beta_k\} \qquad \frac{\text{F-D}}{\text{B-E}}$$

$$= \sum_k \{\lambda_k^{(1)} (\delta_{kk'} \mp \alpha_k^\dagger \alpha_{k'})\alpha_k \ddagger \lambda_k^{(2)} \beta_k^\dagger \beta_k \alpha_{k'}\} \qquad \frac{\text{F-D}}{\text{B-E}}$$

$$= \lambda_{k'}^{(1)} \alpha_{k'} + \mathcal{H}\alpha_{k'}$$

Corollary

If $\mathcal{H}$ is as above, then

$$\text{I.}\quad \alpha_{k'} e^{-i\mathcal{H}t/h} = e^{-i\mathcal{H}t/\hbar}\, \alpha_{k'}\, e^{-i\lambda_{k'}^{(1)} t/\hbar}$$

(1.12)

$$\text{II.}\quad \beta_{k'} e^{-i\mathcal{H}t/\hbar} = e^{-i\mathcal{H}t/\hbar}\, \beta_{k'}\, e^{-i\lambda_{k'}^{(2)} t/\hbar}$$

<u>Proof</u>: We shall prove I only. The proof for II is the same.

$$\alpha_{k'} e^{-i\mathcal{H}t/\hbar} = \sum_{n=0}^{\infty} \frac{(-it/\hbar)^n}{n!} \alpha_{k'} \mathcal{H}^n \quad .$$

But $\alpha_{k'} \mathcal{H}^n = (\mathcal{H}\alpha_{k'} + \lambda_{k'}^{(1)} \alpha_{k'}) \mathcal{H}^{n-1} = \ldots\ldots = (\mathcal{H} + \lambda_{k'}^{(1)})^n \alpha_{k'}$

$$\therefore \quad \alpha_{k'} e^{-i\mathcal{H}t/\hbar} = e^{-i\mathcal{H}t/\hbar} \alpha_{k'} e^{-i\lambda_{k'}^{(1)} t/\hbar} \quad . \qquad \text{Q.E.D.}$$

To get the time evolution of the system, first calculate e.g.:

$$e^{i\mathcal{H}t/\hbar} a_k e^{-i\mathcal{H}t/\hbar}$$

$$= e^{i\mathcal{H}t/\hbar} \sum_{k'} (A^*_{k'k}\alpha_{k'} + C^*_{k'k}\beta_{k'}) e^{-i\mathcal{H}t/\hbar}$$

$$= \sum_{k'} (A^*_{k'k} e^{-i\lambda_{k'}^{(1)} t/\hbar} \alpha_{k'} + C^*_{k'k} e^{-i\lambda_{k'}^{(2)} t/\hbar} \beta_{k'})$$

$$= \sum_{k'} [P_{kk'}(t) a_{k'} + Q_{kk'}(t) b_{k'}] \qquad (1.13)$$

where

$$P_{kk'}(t) = \sum_{k''} (A^*_{k''k} A_{k''k'} e^{-i\lambda_{k''}^{(1)} t/\hbar} + C^*_{k''k} C_{k''k'} e^{-i\lambda_{k''}^{(2)} t/\hbar})$$

$$Q_{kk'}(t) = \sum_{k''} (A^*_{k''k} B_{k''k'} e^{-i\lambda_{k''}^{(1)} t/\hbar} + C^*_{k''k} D_{k''k'} e^{-i\lambda_{k''}^{(2)} t/\hbar}) \, .$$

$$(1.14)$$

and similar expressions for b_k, $a_k^\dagger$, and $b_k^\dagger$. This enables us to calculate the time dependence of

the number density e.g. of particles a in state k:

$$n_k^{(1)}(t) \equiv \mathrm{Tr}(a_k^\dagger a_k \rho_t)/\mathrm{Tr}(\rho_t)$$

$$= \mathrm{Tr}(e^{+i\mathcal{H}t/\hbar}\, a_k^\dagger a_k\, e^{-i\mathcal{H}t/\hbar}\, \rho_o)/\mathrm{Tr}(\rho_t)$$

$$= \mathrm{Tr}\{\sum_{qq'} [P_{kq}^*(t)a_q^\dagger + Q_{kq}^*(t)b_q^\dagger] \times$$

$$\times [P_{kq'}(t)a_{q'} + Q_{kq'}(t)b_{q'}]\rho_o\}/\mathrm{Tr}(\rho_o)$$

$$= \sum_{qq'} \Bigg(P_{kq}^*(t)P_{kq'}(t)\,\frac{\mathrm{Tr}(a_q^\dagger a_{q'} e^{-\beta\mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta\mathcal{H}_o})}$$

$$+ Q_{kq}^*(t)Q_{kq'}(t)\,\frac{\mathrm{Tr}(b_q^\dagger b_{q'} e^{-\beta\mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta\mathcal{H}_o})}$$

$$+ P_{kq}^*(t)Q_{kq'}(t)\,\frac{\mathrm{Tr}(a_q^\dagger b_{q'} e^{-\beta\mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta\mathcal{H}_o})}$$

$$+ Q_{kq}^*(t)P_{kq'}(t)\,\frac{\mathrm{Tr}(b_q^\dagger a_{q'} e^{-\beta\mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta\mathcal{H}_o})} \Bigg) \qquad (1.15)$$

where $\rho_o = e^{-\beta\mathcal{H}_o}$ is the equilibrium statistical operator for $t \leq 0$. The traces can be easily evaluated using the commutation relations (1.9) and the corollary to Theorem 2 (replacing $\mathcal{H}$ by $\mathcal{H}_o$; α,β by a,b; $\lambda^{(i)}$ by $\varepsilon^{(i)}$, and $it/\hbar$ by β). The result is

$$\frac{\mathrm{Tr}(a_k^\dagger a_{k'} e^{-\beta \mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta \mathcal{H}_o})} = \delta_{kk'}\, n^{(o)}(\varepsilon_k^{(1)}) \tag{1.16}$$

$$\frac{\mathrm{Tr}(b_k^\dagger b_{k'} e^{-\beta \mathcal{H}_o})}{\mathrm{Tr}(e^{-\beta \mathcal{H}_o})} = \delta_{kk'}\, n^{(o)}(\varepsilon_k^{(2)}) \tag{1.17}$$

$$\mathrm{Tr}(a_k^\dagger b_{k'} e^{-\beta \mathcal{H}_o}) = \mathrm{Tr}(b_k^\dagger a_{k'} e^{-\beta \mathcal{H}_o}) = 0 \tag{1.18}$$

where

$$n^{(o)}(\varepsilon_k^{(i)}) = \frac{1}{e^{-\beta\mu} e^{\beta\varepsilon_k^{(i)}} \pm 1} \qquad \frac{\text{F-D}}{\text{B-E}}$$

is either the Bose-Einstein or the Fermi-Dirac distribution function, depending on the statistics the operators obey.

Using (1.16)-(1.18), eqn. (1.15) for the time evolution of the number density of the a particles becomes

$$n_k^{(1)}(t) = \sum_{k'} [|P_{kk'}(t)|^2\, n^{(o)}(\varepsilon_{k'}^{(1)}) + |Q_{kk'}(t)|^2\, n^{(o)}(\varepsilon_{k'}^{(2)})] \tag{1.19}$$

The general structure of the coefficients $|P_{kk'}(t)|^2$ and $|Q_{kk'}(t)|^2$ is as follows:

$$\begin{aligned} |P_{kk'}(t)|^2 = \tilde{P}_{kk'} &+ \sum_{q,q'} P'_{kk'qq'} \cos(\lambda_q^{(1)} - \lambda_{q'}^{(2)}) \frac{t}{\hbar} \\ &+ \sum_{q\neq q'} P''_{kk'qq'} \cos(\lambda_{q'}^{(1)} - \lambda_q^{(1)}) \frac{t}{\hbar} \\ &+ \sum_{q\neq q'} P'''_{kk'qq'} \cos(\lambda_{q'}^{(2)} - \lambda_q^{(2)}) \frac{t}{\hbar} \quad . \end{aligned} \tag{1.20}$$

Here $\tilde{P}_{kk'}$ measures the amount of a particles still present in the new steady state as $t \to \infty$. The last three terms give the momentum-state mixing of particles a and b together, and of particles a among themselves, and of b among themselves, respectively.

All these expressions simplify considerably if we assume a constant of field $V_{kk'} = V_o \delta_{kk'}$, because then no mixing of momentum states can occur. In particular, we get

$$n_k^{(1)}(t) = n^{(o)}(\varepsilon_k^{(1)}) + \frac{1}{2} \frac{V_o^2}{V_o^2 + \xi_k^2} [1 - \cos(\lambda_k^{(1)} - \lambda_k^{(2)}) \frac{t}{\hbar}] \times [n^{(o)}(\varepsilon_k^{(2)}) - n^{(o)}(\varepsilon_k^{(1)})] \tag{1.21}$$

with

$$\lambda_k^{(1)} - \lambda_k^{(2)} = 2(V_o^2 + \xi_k^2)^{\frac{1}{2}}$$

$$\xi_k = \frac{1}{2}(\varepsilon_k^{(1)} - \varepsilon_k^{(2)}) \quad .$$

The single particle energies are given by

$$\begin{aligned} \varepsilon_k^{(1)} &= \frac{\hbar^2 k^2}{2m_1} \\ \varepsilon_k^{(2)} &= \frac{\hbar^2 k^2}{2m_2} + \varepsilon_o \end{aligned} \tag{1.22}$$

where ε_o is the threshold energy of formation of particles b.

We now specialize to a system consisting initially at $t \leq 0$ of only particles a in equilibrium at a temperature T, a constant external field V_o being switched on spontaneously at $t = 0$. We then get for the density of particles a in the thermodynamic limit

$$
\begin{aligned}
\rho_1(t) &= \frac{1}{V} \sum_k n_k^{(1)}(t) \\
&= \frac{1}{(2\pi)^3} \int d^3k \; n_k^{(1)}(t) \\
&= \rho_1(t=0) - \frac{1}{(2\pi)^3} \int d^3k \; \frac{1}{2} \frac{V_o^2}{V_o^2+\xi_k^2} \, n^{(o)}(\varepsilon_k^{(1)}) \\
&\quad \times [1 - \cos(\lambda_k^{(1)} - \lambda_k^{(2)}) \tfrac{t}{\hbar}] \; . \qquad (1.23)
\end{aligned}
$$

Making the substitutions $\delta = \frac{\mu}{m_1} \frac{V_o}{kT}$, $r = \frac{\varepsilon_o}{V_o}$,

$$\tau = \frac{2V_o t}{\hbar} , \quad \mu = \frac{m_1 m_2}{m_2 - m_1} \quad \text{and} \quad x^2 = \frac{\hbar^2}{2\mu V_o} k^2 \qquad (1.24)$$

and assuming T is high enough to use Maxwell-Boltzmann statistics for simplicity, the time evolution equation becomes

$$\frac{\rho_1(t)}{\rho_1(0)} = 1 - \frac{2}{\sqrt{\pi}} \delta^{3/2} \int_0^\infty \frac{x^2 dx}{1+\frac{1}{4}(x^2-r)^2} e^{-\delta x^2} \{1 - \cos \tau [1+\tfrac{1}{4}(x^2-r)^2]^{\frac{1}{2}}\} \qquad (1.25)$$

Several examples of this equation are plotted in Fig. 1.1, on the next page. A discussion of this diagram, taken from Reference [1], will be given in Chapter IV.

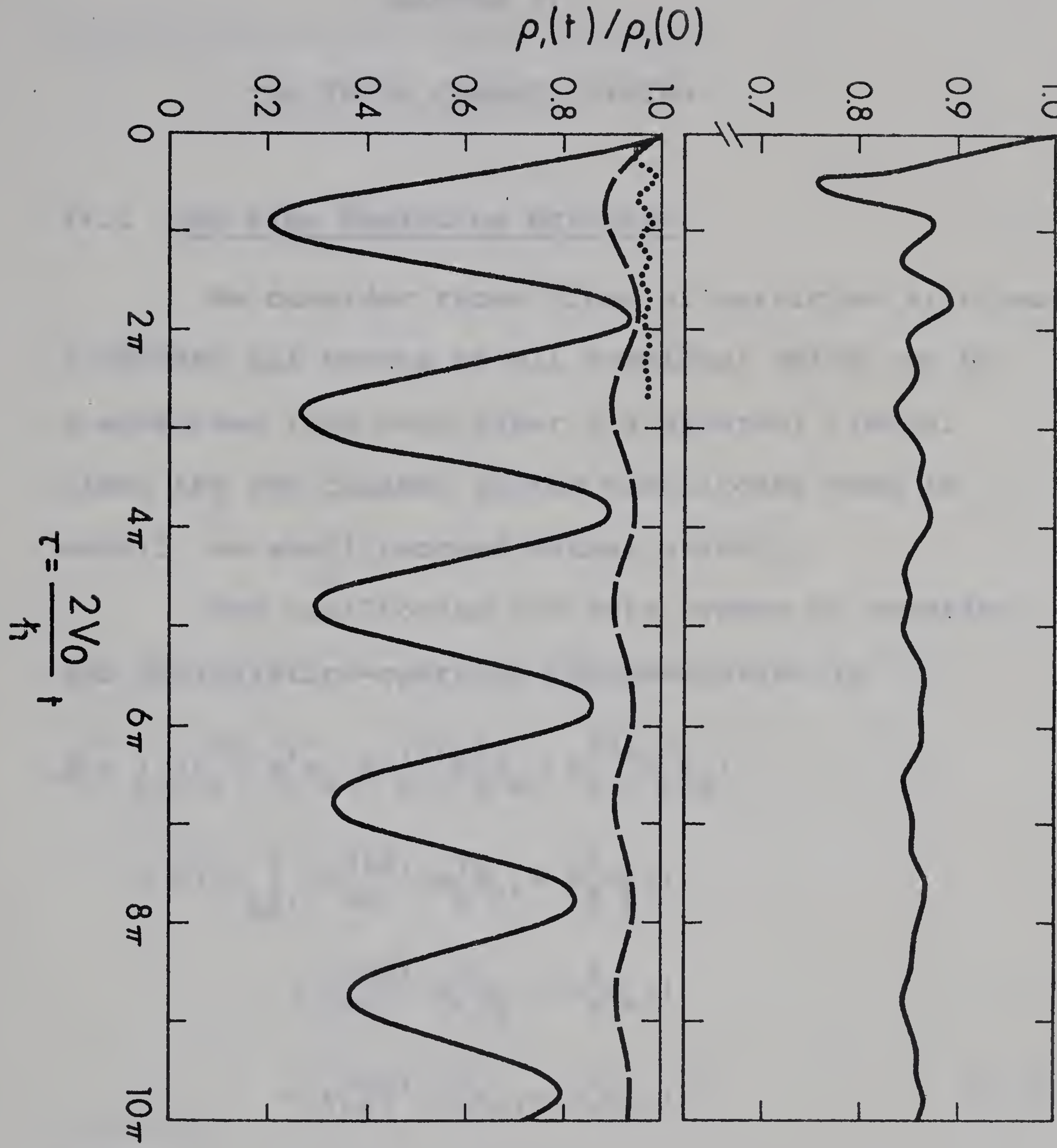


Fig. 1.1. Typical examples of density oscillations in the two channel model. (Taken from [1]).
Upper graph: δ = 1.; r = 5.
Lower graph: dotted line: δ = 1.; r = 10.
dashed line: δ = 0.1; r = 1.
solid line: δ = 1.; r = 1.

CHAPTER II

THE THREE CHANNEL SYSTEM

§2.1 The Time Evolution Equation

We consider three kinds of particles a, b and c (either all bosons or all fermions) which can be transformed into each other via external fields. Since the two channel system was already done in detail, we shall proceed rather quickly.

The hamiltonian for this system in creation- and annihilation-operator representation is

$$\begin{aligned} \mathcal{H} = & \sum_k (\varepsilon_k^{(1)} a_k^\dagger a_k + \varepsilon_k^{(2)} b_k^\dagger b_k + \varepsilon_k^{(3)} c_k^\dagger c_k) \\ & + \theta(t) \sum_{kk'} \{V_{kk'}^{(12)} (a_k^\dagger b_{k'} + b_k^\dagger a_{k'}) \\ & \qquad + V_{kk'}^{(13)} (a_k^\dagger c_{k'} + c_k^\dagger a_{k'}) \\ & \qquad + V_{kk'}^{(23)} (b_k^\dagger c_{k'} + c_k^\dagger b_{k'})\} \end{aligned} \tag{2.1}$$

where

$$V_{kk'}^{(ij)} = \frac{1}{V} \int dx \, V^{(ij)}(x) e^{i(k-k')\cdot x}$$

As before, if $V^{(ij)}(x) = V^{(ij)}$ is a constant interaction then $V_{kk'}^{(ij)} = V^{(ij)} \delta_{kk'}$.

<u>Diagonalization and exact solution</u>

In matrix notation

$$\mathcal{H} = (a_k^\dagger | b_k^\dagger | c_k^\dagger) \underset{\approx}{\varepsilon} \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \tag{2.2}$$

where

$$\underset{\approx}{\varepsilon} = \begin{pmatrix} E^{(1)} & V^{(12)} & V^{(13)} \\ V^{(12)\dagger} & E^{(2)} & V^{(23)} \\ V^{(13)\dagger} & V^{(23)\dagger} & E^{(3)} \end{pmatrix} \tag{2.3}$$

and $E^{(i)}$ and $V^{(ij)}$ are submatrices with elements $E^{(i)}_{kk'} = \varepsilon^{(i)}_k \delta_{kk'}$ and $V^{(ij)}_{kk'}$ respectively. The hamiltonian eqn. (2.2) can be diagonalized with a unitary matrix

$$U = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & J \end{pmatrix} \quad . \tag{2.4}$$

That is:

$$\mathcal{H} = (\alpha_k^\dagger | \beta_k^\dagger | \gamma_k^\dagger) \underset{\approx}{\Lambda} \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \end{pmatrix} \tag{2.5}$$

where

$$\underset{\approx}{\Lambda} = \begin{pmatrix} \Lambda^{(1)} & & \\ & \Lambda^{(2)} & \\ & & \Lambda^{(3)} \end{pmatrix} = U \underset{\approx}{\varepsilon} U^{-1} \tag{2.6}$$

and $\Lambda^{(i)}$ has elements $\Lambda^{(i)}_{kk'} = \delta_{kk'}\lambda^{(i)}_k$. We introduce quasi-particles by a linear transformation of the annihilation operators

$$\begin{aligned}
\alpha_k &= \sum_{k'} (A_{kk'}a_{k'} + B_{kk'}b_{k'} + C_{kk'}c_{k'}) \\
\beta_k &= \sum_{k'} (D_{kk'}a_{k'} + E_{kk'}b_{k'} + F_{kk'}c_{k'}) \qquad (2.7) \\
\gamma_k &= \sum_{k'} (G_{kk'}a_{k'} + H_{kk'}b_{k'} + J_{kk'}c_{k'})
\end{aligned}$$

with inverse transformations

$$\begin{aligned}
a_k &= \sum_{k'} (A^*_{k'k}\alpha_{k'} + D^*_{k'k}\beta_{k'} + G^*_{k'k}\gamma_{k'}) \\
b_k &= \sum_{k'} (B^*_{k'k}\alpha_{k'} + E^*_{k'k}\beta_{k'} + H^*_{k'k}\gamma_{k'}) \qquad (2.8) \\
c_k &= \sum_{k'} (C^*_{k'k}\alpha_{k'} + F^*_{k'k}\beta_{k'} + J^*_{k'k}\gamma_{k'})
\end{aligned}$$

and similar equations for the creation operators. Because of the unitarity of U, the quasi-particle operators again satisfy the same commutation relations as the particle operators. That is, if

$$\begin{aligned}
[a_k, a^\dagger_{k'}]_\pm &= \delta_{kk'} \\
[a_k, b^\dagger_{k'}]_\pm &= 0 \qquad \frac{F-D}{B-E} \qquad (2.9) \\
[a_k, a_{k'}]_\pm &= 0
\end{aligned}$$

etc., then

$$[\alpha_k, \alpha_{k'}^{\dagger}]_{\pm} = \delta_{kk'}$$

$$[\alpha_k, \beta_{k'}^{\dagger}]_{\pm} = 0 \qquad \frac{F-D}{B-E} \tag{2.10}$$

$$[\alpha_k, \alpha_{k'}]_{\pm} = 0$$

etc.

To get the time evolution of the system, use Theorem 2 and its Corollary from Chapter One to first calculate e.g.

$$e^{i\mathcal{H}t/\hbar}\, a_k\, e^{-i\mathcal{H}t/\hbar}$$

$$= e^{i\mathcal{H}t/\hbar} \sum_{k'} (A^*_{k'k}\alpha_{k'} + D^*_{k'k}\beta_{k'} + G^*_{k'k}\gamma_{k'})\, e^{-i\mathcal{H}t/\hbar}$$

$$= \sum_{k'} (A^*_{k'k} e^{-i\lambda^{(1)}_{k'} t/\hbar} \alpha_{k'} + D^*_{k'k} e^{-i\lambda^{(2)}_{k'} t/\hbar} \beta_{k'}$$

$$+ G^*_{k'k} e^{-i\lambda^{(3)}_{k'} t/\hbar} \gamma_{k'})$$

$$= \sum_{k'} [P_{kk'}(t)a_{k'} + Q_{kk'}(t)b_{k'} + R_{kk'}(t)c_{k'}] \tag{2.11}$$

where

$$P_{kk''}(t) = \sum_{k'} (A^*_{k'k}A_{k'k''} e^{-i\lambda^{(1)}_{k'} t/\hbar} + D^*_{k'k}D_{k'k''} e^{-i\lambda^{(2)}_{k'} t/\hbar}$$

$$+ G^*_{k'k}G_{k'k''} e^{-i\lambda^{(3)}_{k'} t/\hbar}) \tag{2.12}$$

$$Q_{kk''}(t) = \sum_{k'} (A^*_{k'k}B_{k'k''} e^{-i\lambda^{(1)}_{k'} t/\hbar} + D^*_{k'k}E_{k'k''} e^{-i\lambda^{(2)}_{k'} t/\hbar}$$

$$+ G^*_{k'k}H_{k'k''} e^{-i\lambda^{(3)}_{k'} t/\hbar}) \tag{2.13}$$

$$R_{kk''}(t) = \sum_{k'} (A^*_{k'k} C_{k'k''} e^{-i\lambda^{(1)}_{k'} t/\hbar} + D^*_{k'k} F_{k'k''} e^{-i\lambda^{(2)}_{k'} t/\hbar}$$

$$+ G^*_{k'k} J_{k'k''} e^{-i\lambda^{(3)}_{k'} t/\hbar}) \qquad (2.14)$$

and similar expressions for b_k, c_k, $a^\dagger_k$, $b^\dagger_k$ and $c^\dagger_k$. This enables us to calculate the time dependence of the number density e.g. of particles a in state k:

$$n^{(1)}_k(t) \equiv \frac{\mathrm{Tr}(a^\dagger_k a_k \rho_t)}{\mathrm{Tr}(\rho_t)} = \frac{\mathrm{Tr}(e^{i\mathcal{H}t/\hbar} a^\dagger_k a_k e^{-i\mathcal{H}t/\hbar} \rho_o)}{\mathrm{Tr}(\rho_o)}$$

$$= \sum_{k'} [|P_{kk'}(t)|^2 n^{(o)}(\varepsilon^{(1)}_{k'}) + |Q_{kk'}(t)|^2 n^{(o)}(\varepsilon^{(2)}_{k'})$$

$$+ |R_{kk'}(t)|^2 n^{(o)}(\varepsilon^{(3)}_{k'})] \qquad (2.15)$$

where $\rho_o = e^{-\beta\mathcal{H}_o}$ is the equilibrium statistical operator for $t \leq 0$ and $n^{(o)}(\varepsilon^{(i)}_{k'})$ are the B-E or F-D distribution functions, depending on the statistics the operators obey.

<u>No momentum mixing</u>

All these expressions simplify considerably if we consider a constant field $V^{(ij)}_{kk'} = V^{(ij)}\delta_{kk'}$ because then no momentum mixing occurs. In particular, each of the sub-matrices of U, i.e. A,B,...J are diagonal. This is proven in the following three theorems.

Theorem 3

If A, B, C, D are any general block matrices with A and D square, then [2]:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \cdot |D - CA^{-1}B|$$

Proof:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & O \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & O \\ O & D-CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ O & I \end{pmatrix}$$

$$\rightarrow \begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1 \cdot |A(D-CA^{-1}B)| \cdot 1$$

Q.E.D.

Theorem 4

If A,B,...J are square diagonal submatrices, then

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & J \end{vmatrix} = |A(EJ-FH) - B(DJ-FG) + C(DH-EG)|$$

Proof: Use Theorem 3 with the matrix partitioned as shown:

$$\left|\begin{array}{c|cc} A & B & C \\ \hline D & E & F \\ G & H & J \end{array}\right| = |A| \cdot \left|\begin{pmatrix} E-DA^{-1}B & F-DA^{-1}C \\ H-GA^{-1}B & J-GA^{-1}C \end{pmatrix}\right|$$

$$= |A(EJ-FH) - B(DJ-FG) + C(DH-EG)| \ .$$

Q.E.D.

Theorem 5

If $\underset{\approx}{\varepsilon}$ is as given in eqn. (2.3), then its eigenvalues are found by solving the equation $|\lambda I - \underset{\approx}{\varepsilon}| = 0$. In the transformation matrix U, each submatrix A,B,...J is diagonal if $V^{(ij)}_{kk'} = V^{(ij)}\delta_{kk'}$.

Proof: Using Theorem 4,

$$0 = |\lambda I - \underset{\approx}{\varepsilon}| = |F(\lambda)| \tag{2.16}$$

where $F(\lambda)$ is a diagonal matrix with elements

$$\begin{aligned} F(\lambda)_{kk'} = \delta_{kk'}\{&\lambda^3 - \lambda^2(\varepsilon_k^{(1)} + \varepsilon_k^{(2)} + \varepsilon_k^{(3)}) \\ &+ \lambda(\varepsilon_k^{(1)}\varepsilon_k^{(2)} + \varepsilon_k^{(1)}\varepsilon_k^{(3)} + \varepsilon_k^{(2)}\varepsilon_k^{(3)} - V^{(23)^2} - V^{(12)^2} - V^{(13)^2}) \\ &- (\varepsilon_k^{(1)}\varepsilon_k^{(2)}\varepsilon_k^{(3)} + 2V^{(12)}V^{(13)}V^{(23)} - \varepsilon_k^{(1)}V^{(23)^2} - \varepsilon_k^{(2)}V^{(13)^2} \\ &- \varepsilon_k^{(3)}V^{(12)^2})\}. \end{aligned} \tag{2.17}$$

Expanding the determinant

$$0 = |F(\lambda)| = \prod_k F(\lambda)_{kk}$$

$$\longleftrightarrow \quad F(\lambda)_{kk} = 0 \qquad \text{for all } k.$$

This third order equation in λ has the solutions $\lambda_k^{(i)}$, $i = 1,2,3$. The eigenvalues are arranged in the order given in (2.6) and $\Lambda^{(i)}$ has the elements $\Lambda^{(i)}_{kk'} = \lambda_k^{(i)}\delta_{kk'}$.

The columns of U^{-1} are formed from the normalized column vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where

$$x \equiv \begin{pmatrix} \vdots \\ x_k \\ \vdots \end{pmatrix}$$

etc., which are obtained by solving the equation

$$[\lambda I - \underset{\approx}{\varepsilon}] \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.18}$$

If $\lambda = \lambda_k^{(i)}$, we get the following system

$$(\lambda_k^{(i)} - \varepsilon_\ell^{(1)}) x_\ell - V^{(12)} y_\ell - V^{(13)} z_\ell = 0$$

$$-V^{(12)} x_\ell + (\lambda_k^{(i)} - \varepsilon_\ell^{(2)}) y_\ell - V^{(23)} z_\ell = 0 \tag{2.19}$$

$$-V^{(13)} x_\ell - V^{(23)} y_\ell + (\lambda_k^{(i)} - \varepsilon_\ell^{(3)}) z_\ell = 0$$

for each ℓ. There are two cases: $\ell \neq k$ and $\ell = k$.

$\underline{\ell \neq k}$ Then this set of 3 equations (for a given ℓ) has determinant $\neq 0$ since $\lambda_k^{(i)}$ and $\varepsilon_\ell^{(i)}$ are not algebraically related.

$\therefore \quad x_\ell = y_\ell = z_\ell = 0$ for all $\ell \neq k$.

$\underline{\ell = k}$ Since the above equations all had det. $\neq 0$ and yet $|\lambda I - \underset{\approx}{\varepsilon}| = 0$, so the three equations for $\ell = k$ must have det.= 0.

$\therefore$ at least one of x_k, y_k, z_k is not equal to zero.

Thus

$$A_{kk'} = \delta_{kk'} x_k \quad ,$$

$$B_{kk'} = \delta_{kk'} y_k \quad ,$$

$$C_{kk'} = \delta_{kk'} z_k \quad ,$$

all formed with $\lambda = \lambda_k^{(1)}$. The other submatrices similarly are found to be diagonal.

Q.E.D.

If we furthermore assume that there are only particles a present at time t = 0, then the time dependence of the number density of the a, b, and c particles is

$$n_k^{(i)}(t) = |P_{kk}^{(i)}(t)|^2 \, n^{(o)}(\varepsilon_k^{(1)}) \qquad i = 1,2,3 \tag{2.20}$$

where

$$|P_{kk}^{(1)}(t)|^2 = 1-2A^2D^2(1-f_1)-2A^2G^2(1-f_2)-2D^2G^2(1-f_3) \tag{2.21}$$

$$|P_{kk}^{(2)}(t)|^2 = 0-2ABDE(1-f_1)-2ABGH(1-f_2)-2DEGH(1-f_3) \tag{2.22}$$

$$|P_{kk}^{(3)}(t)|^2 = 0-2ACDF(1-f_1)-2ACGJ(1-f_2)-2DFGJ(1-f_3) \tag{2.23}$$

with

$$\begin{aligned} f_1 &\equiv \cos(\lambda_k^{(2)} - \lambda_k^{(1)}) \frac{t}{\hbar} \\ f_2 &\equiv \cos(\lambda_k^{(3)} - \lambda_k^{(1)}) \frac{t}{\hbar} \\ f_3 &\equiv \cos(\lambda_k^{(3)} - \lambda_k^{(2)}) \frac{t}{\hbar} \end{aligned} \tag{2.24}$$

and the unitary matrix's submatrices A,B,...J have subscripts k,k. i.e.: $A \equiv A_{kk}$, etc. Note that

$$|P_{kk}^{(i)}(t=0)|^2 = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i = 2,3 \end{cases}$$

as was expected, and that by the unitarity of U,

$$\sum_{i=1}^{3} |P_{kk}^{(i)}(t)|^2 = 1 .$$

i.e.: Conservation of total number is guaranteed by the unitarity of U.

We get for the density of particles a, b, and c in the thermodynamic limit

$$\begin{aligned} \rho_i(t) &= \frac{1}{V} \sum_k n_k^{(i)}(t) \\ &= \frac{1}{(2\pi)^3} \int d^3k \; n_k^{(i)}(t) \\ &= \frac{1}{(2\pi)^3} \int d^3k \, |P_{kk}^{(i)}(t)|^2 \; n^{(o)}(\varepsilon_k^{(1)}). \end{aligned} \quad (2.25)$$

The single particle energies are given by

$$\varepsilon_k^{(1)} = \frac{\hbar^2 k^2}{2m_1} \quad ,$$

$$\varepsilon_k^{(2)} = \frac{\hbar^2 k^2}{2m_2} + \varepsilon_2 \; ,$$

$$\varepsilon_k^{(3)} = \frac{\hbar^2 k^2}{2m_3} + \varepsilon_3 \; ,$$

where ε_2, ε_3, are threshold energies for the formation of b, c, particles respectively. To make the final time evolution equation dimensionless, write $\underset{\approx}{\varepsilon}$ as

$$\underset{\approx}{\varepsilon} = V^{(12)} \begin{pmatrix} S^{(1)} & I & U^{(1)} \\ I & S^{(2)} & U^{(2)} \\ U^{(1)} & U^{(2)} & S^{(3)} \end{pmatrix} \equiv V^{(12)} \left(\underset{\approx}{\varepsilon} / V^{(12)} \right) \tag{2.26}$$

where the submatrices have the elements

$$S^{(1)}_{kk'} = \frac{\hbar^2 k^2}{2m_1 V_{12}} \delta_{kk'}$$

$$S^{(i)}_{kk'} = \left(\frac{\hbar^2 k^2}{2m_i V_{12}} + \frac{\varepsilon_i}{V_{12}} \right) \delta_{kk'} \qquad i = 2,3 \tag{2.27}$$

$$I_{kk'} = \delta_{kk'}$$

$$U^{(i)}_{kk'} = \frac{V^{(i3)}}{V^{(12)}} \delta_{kk'} \qquad i = 1,2$$

$\underset{\approx}{\varepsilon}$ is diagonalized by the orthogonal matrix U, given in (2.4). i.e.:

$$\Lambda = U \underset{\approx}{\varepsilon} U^{-1} = V^{(12)} \left[U \left(\underset{\approx}{\varepsilon} / V^{(12)} \right) U^{-1} \right]$$

$$= V^{(12)} P = V^{(12)} \begin{pmatrix} P^{(1)} & & \\ & P^{(2)} & \\ & & P^{(3)} \end{pmatrix} \tag{2.28}$$

Using the substitutions

$$x^2 = \frac{\hbar^2 k^2}{2m_1 V^{(12)}} \qquad \delta = \beta V^{(12)} \qquad \tau = \frac{2V^{(12)}t}{\hbar}$$

$$\frac{\lambda_k^{(\ell)}}{V^{(12)}} = P_k^{(\ell)} \qquad U_1 = \frac{V^{(13)}}{V^{(12)}} \qquad U_2 = \frac{V^{(23)}}{V^{(12)}} \tag{2.29}$$

$$r_2 = \frac{\varepsilon_2}{V^{(12)}} \qquad r_3 = \frac{\varepsilon_3}{V^{(12)}}$$

(and dropping the subscripts of the submatrices of U, and the P's) and using Maxwell-Boltzmann statistics for simplicity, the time evolution equation (2.25) becomes

$$\begin{pmatrix} \frac{\rho_1(t)}{\rho_1(0)} \\ \frac{\rho_2(t)}{\rho_1(0)} \\ \frac{\rho_3(t)}{\rho_1(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{8}{\sqrt{\pi}}\,\delta^{3/2} \int_0^\infty x^2 dx\, e^{-\delta x^2} \times$$

$$\times \left\{ A \begin{pmatrix} A \\ B \\ C \end{pmatrix} D \begin{pmatrix} D \\ E \\ F \end{pmatrix} \left(1 - \cos \frac{P_2 - P_1}{2}\tau\right) + A \begin{pmatrix} A \\ B \\ C \end{pmatrix} G \begin{pmatrix} G \\ H \\ J \end{pmatrix} \left(1 - \cos \frac{P_3 - P_1}{2}\tau\right) + D \begin{pmatrix} D \\ E \\ F \end{pmatrix} G \begin{pmatrix} G \\ H \\ J \end{pmatrix} \left(1 - \cos \frac{P_3 - P_2}{2}\tau\right) \right\} \tag{2.30}$$

§2.2 Illustration Examples

Case I Altering the Energy Scale

The frequency of any oscillations in the three channel model is proportional to the energies involved in the model. To show this, suppose all energies involved are multiplied by a factor q. Noting that all energies are rationalized in terms of $V^{(12)}$, we must make the following changes in eqns. (2.26) to (2.30): $\underset{\approx}{\xi} \to q\underset{\sim}{\xi}$, (eq. $V^{(12)} \to qV^{(12)}$) $\delta \to q\delta$, $x^2 \to x^2$, $\tau \to q\tau$, and U and P_i, $i = 1,2,3$ are unchanged. Eqn. (2.30) becomes

$$\frac{\rho_i(t)}{\rho_1(0)} = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{8}{\sqrt{\pi}}(q\delta)^{3/2} \int_0^\infty x^2\, dx\, e^{-q\delta x^2} \times$$

$$\times \left[A \begin{pmatrix}A\\B\\C\end{pmatrix} D \begin{pmatrix}D\\E\\F\end{pmatrix} \left(1 - \cos \frac{P_2 - P_1}{2} q\tau\right) + \ldots \right] \quad (2.31)$$

Thus multiplying the energies by q causes the oscillations to be q times as fast.

Equal Masses and Undamped Oscillations

If $m_1 = m_2 = m_3$, then using the definitions of (2.29) we get that

$$\frac{1}{V^{(12)}} \underset{\approx}{\varepsilon} = \begin{pmatrix} 0 & 1 & U_1 \\ 1 & r_2 & U_2 \\ U_1 & U_2 & r_3 \end{pmatrix} + x^2 I \tag{2.32}$$

where $I_{ij} = \delta_{ij}$. The matrix of rationalized eigenvalues is $\frac{1}{V^{(12)}} \Lambda = P$ where

$$P = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix} = U \begin{pmatrix} 0 & 1 & U_1 \\ 1 & r_2 & U_2 \\ U_1 & U_2 & r_3 \end{pmatrix} U^{-1} + x^2 I$$

$$= \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} + \begin{pmatrix} x^2 & & \\ & x^2 & \\ & & x^2 \end{pmatrix} \quad . \tag{2.33}$$

where μ_1, μ_2, μ_3 and U are constant in x. The integrals (2.30) for the particle densities are then trivial. Note that δ drops out i.e.: there is no temperature dependence. We get

$$\frac{\rho_i(t)}{\rho_1(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 2 \left[A \begin{pmatrix} A \\ B \\ C \end{pmatrix} D \begin{pmatrix} D \\ E \\ F \end{pmatrix} (1 - \cos \omega_1 \tau) \right.$$

$$+ A \begin{pmatrix} A \\ B \\ C \end{pmatrix} G \begin{pmatrix} G \\ H \\ J \end{pmatrix} (1 - \cos \omega_2 \tau)$$

$$\left. + D \begin{pmatrix} D \\ E \\ F \end{pmatrix} G \begin{pmatrix} G \\ H \\ J \end{pmatrix} (1 - \cos \omega_3 \tau) \right] \tag{2.34}$$

where

$$\omega_1 = \frac{\mu_2-\mu_1}{2} \qquad \omega_2 = \frac{\mu_3-\mu_1}{2} \qquad \omega_3 = \frac{\mu_3-\mu_2}{2} \; . \quad (2.35)$$

Thus the oscillations are undamped. This is not true in general because usually the frequencies ω_i are functions of x, resulting in damping.

Case II Altering the $V^{(23)}$ interaction energy

We set $m_1 = m_2 = m_3$, $r_2 = r_3 = 0$ and $V^{(12)}=V^{(13)}$. Thus the a, b and c particles are chemically identical i.e.: the same masses and no threshold energies between them. $V^{(ij)}$ is a measure of the probability of an i into j or a j into i particle transformation. Since $V^{(12)} = V^{(13)}$ and initially all particles are of type a (we shall also say they are all in "state 1"), we expect that $\rho_2(t) = \rho_3(t)$.

From Case I, we guess that the oscillation frequency will increase as $V^{(23)}$ increases. Physically this is reasonable since the larger $V^{(23)}$ is, the more likely transitions between states 2 and 3 are, and hence the faster the oscillations. The frequency can be found exactly:

$$\frac{1}{V^{(12)}} \underset{\approx}{\xi} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & U_2 \\ 1 & U_2 & 0 \end{pmatrix} + x^2 I = M + x^2 I \; . \qquad (2.36)$$

Solving $|\mu I-M| = 0$ we get

$$\mu_1 = -U_2 ,$$

$$\mu_{2,3} = \frac{U_2}{2} \pm \sqrt{\left(\frac{U_2}{2}\right)^2 + 2} .$$

The eigenvector for μ_1 is $(0, +1/\sqrt{2}, -1/\sqrt{2})$. Thus $A = 0$ and the only frequency that enters is

$$\omega_3 = \frac{\mu_3-\mu_2}{2} = \sqrt{\left(\frac{U_2}{2}\right)^2 + 2} . \qquad (2.37)$$

As can be seen from Fig. 2.1, the amplitude of the oscillations also decrease as $V^{(23)}$ increases. This coupling of amplitude and frequency is typical for oscillations in non-linear systems.

Case III Altering the r_3 threshold energy

We again assume that $m_1 = m_2 = m_3$ so the time evolution of densities is governed by eqn. (2.34). Furthermore, let $r_2 = 0$ and $V^{(12)} = V^{(13)} = V^{(23)}$. Then

$$\frac{1}{V^{(12)}} \underset{\approx}{\varepsilon} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & r_3 \end{pmatrix} + x^2 I = M + x^2 I . \qquad (2.38)$$

Solving $|\mu I-M| = 0$ we get $\mu_1 = -1$, $\mu_{2,3} = \frac{r_3+1 \pm \sqrt{(r_3-1)^2+8}}{2}$.

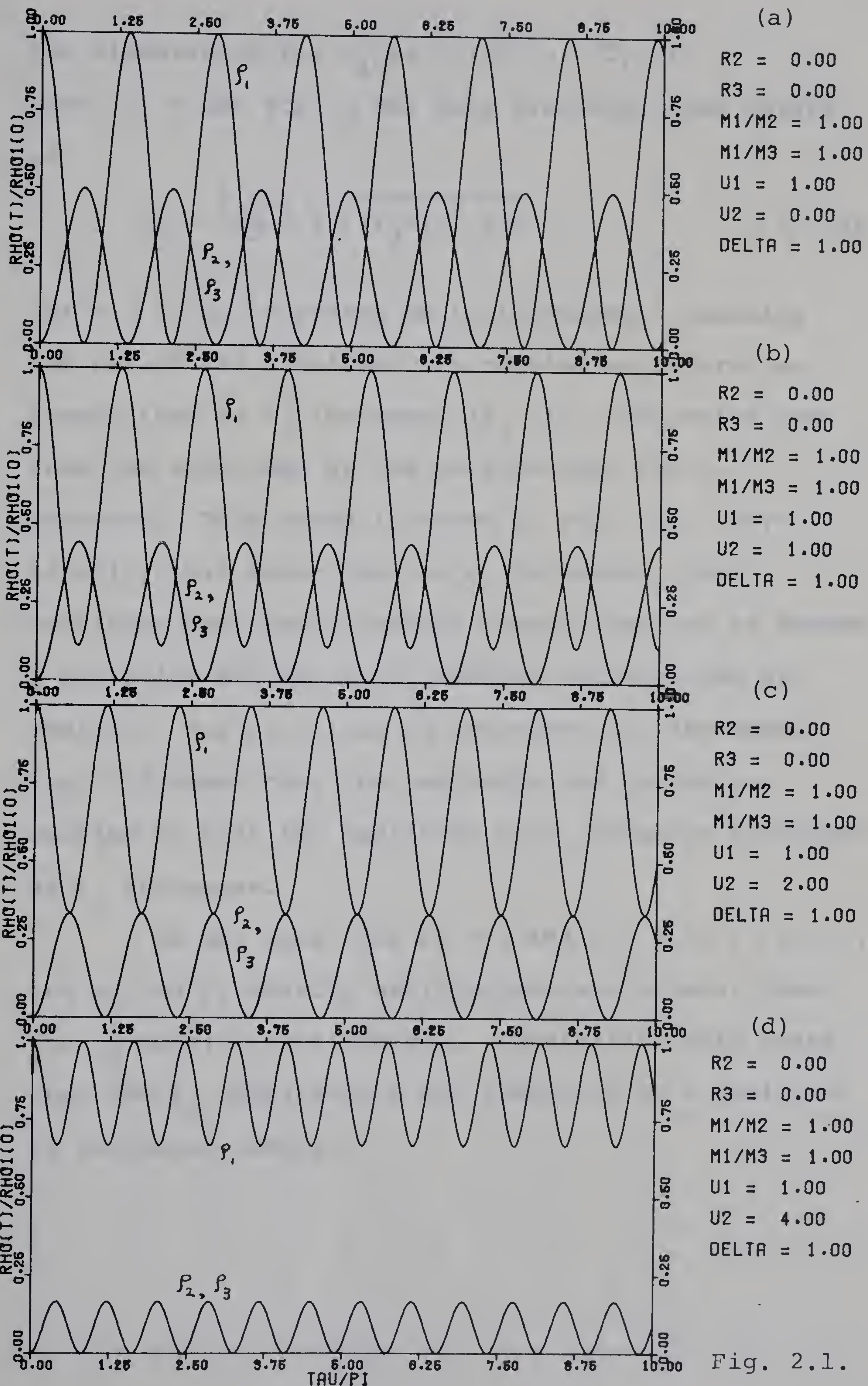

(a)
R2 = 0.00
R3 = 0.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 0.00
DELTA = 1.00
(b)
R2 = 0.00
R3 = 0.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00
(c)
R2 = 0.00
R3 = 0.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 2.00
DELTA = 1.00
(d)
R2 = 0.00
R3 = 0.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 4.00
DELTA = 1.00
RHO(T)/RHO1(0)
TAU/PI
ρ1
ρ2, ρ3

Fig. 2.1.

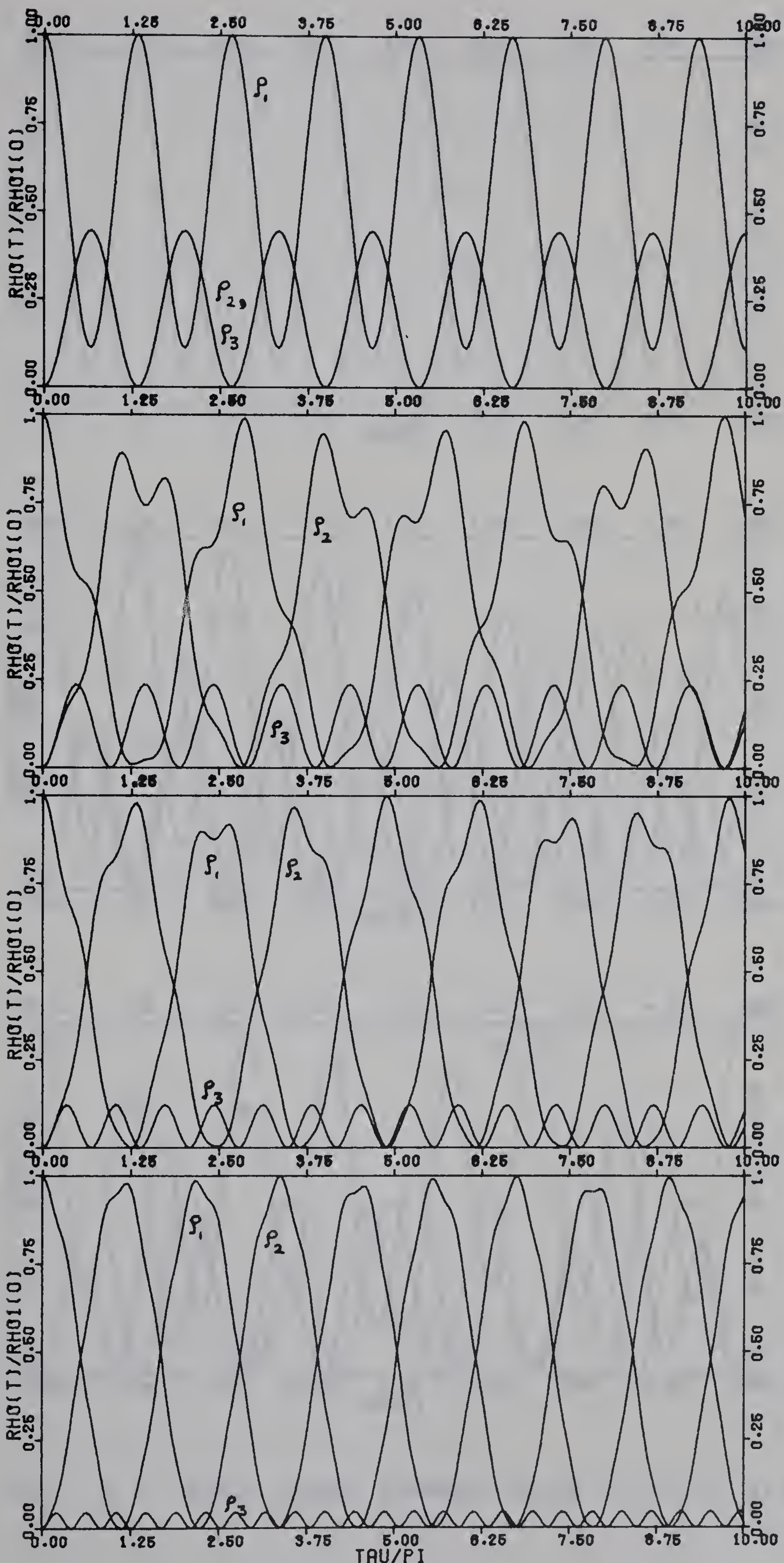
The eigenvector for μ_1 is $(1/\sqrt{2}, -1/\sqrt{2}, 0)$.

Thus $C = 0$ and for ρ_3 the only frequency that enters is

$$\omega_3 = \frac{\mu_3-\mu_2}{2} = \sqrt{(r_3-1)^2 + 8} \quad . \tag{2.39}$$

For $r_3 > 1$, ω_3 increases as r_3 increases. Assuming the period and amplitude are coupled as before, we expect that as r_3 increases $(r_3 > 1)$, the period and thus the amplitude of the oscillations for ρ_3 decrease. This trend is shown in Fig. 2.2. Physically, this means that as r_3 increases, less particles have the threshold energy required to become C particles and so the ρ_3 density oscillations are smaller. For $r_3 < 1$; as r_3 increases, ω_3 decreases. Fig. 2.3 shows that the amplitude and period are coupled so that the amplitude of ρ_3 actually increases as r_3 increases.

In any case, for $r_2 = 0$ and $r_3 > 1$ or $0 < r_3 < 1$, the ρ_1 and ρ_2 density oscillations are greater than the ρ_3 density oscillations. Physically, this means that the ρ_3 oscillations are inhibited by a positive r_3 threshold energy.



(a)

R2 = 0.00
R3 = 0.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00

(b)

R2 = 0.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00

(c)

R2 = 0.00
R3 = 6.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00

(d)

R2 = 0.00
R3 = 10.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00

Fig. 2.2.

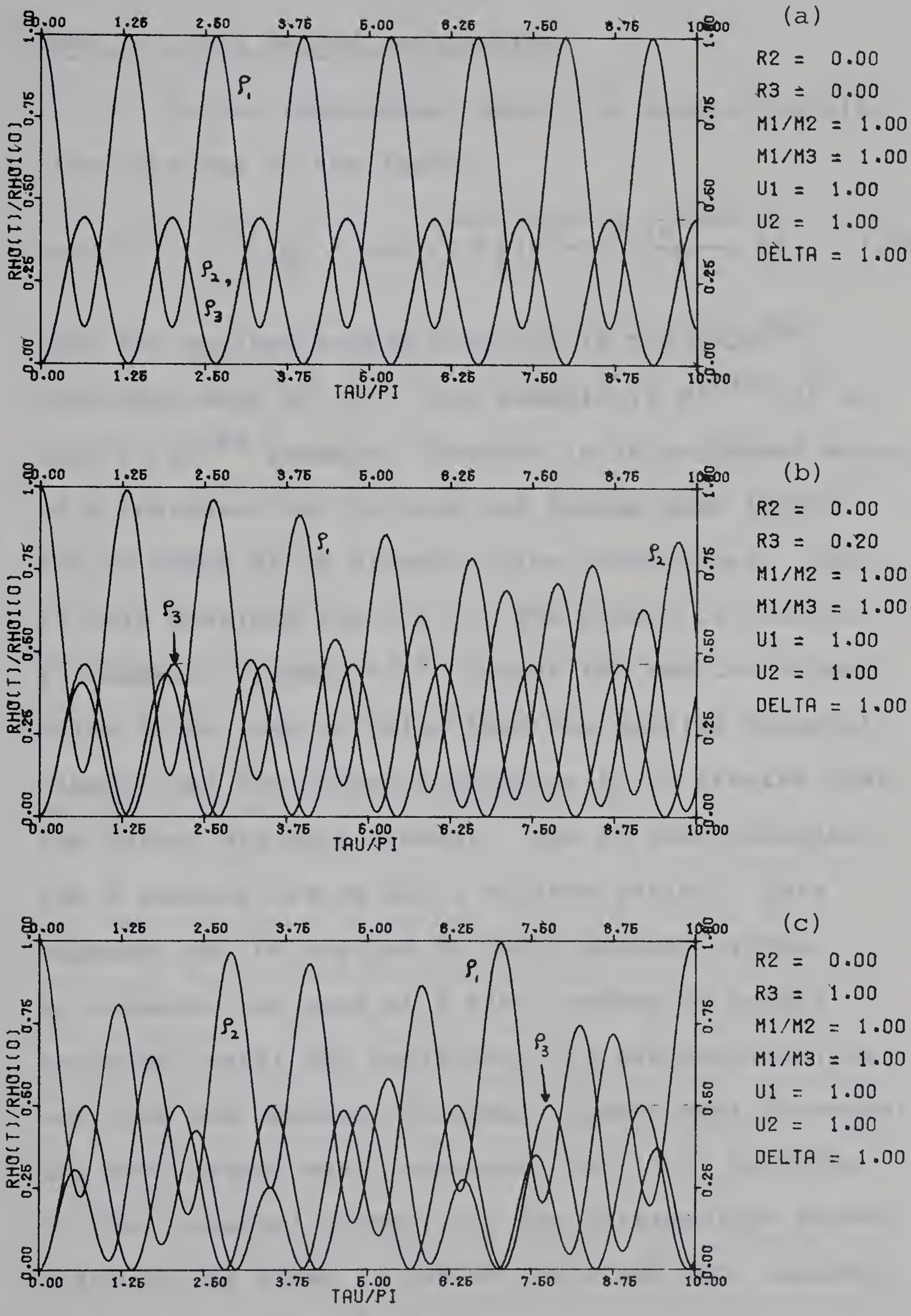
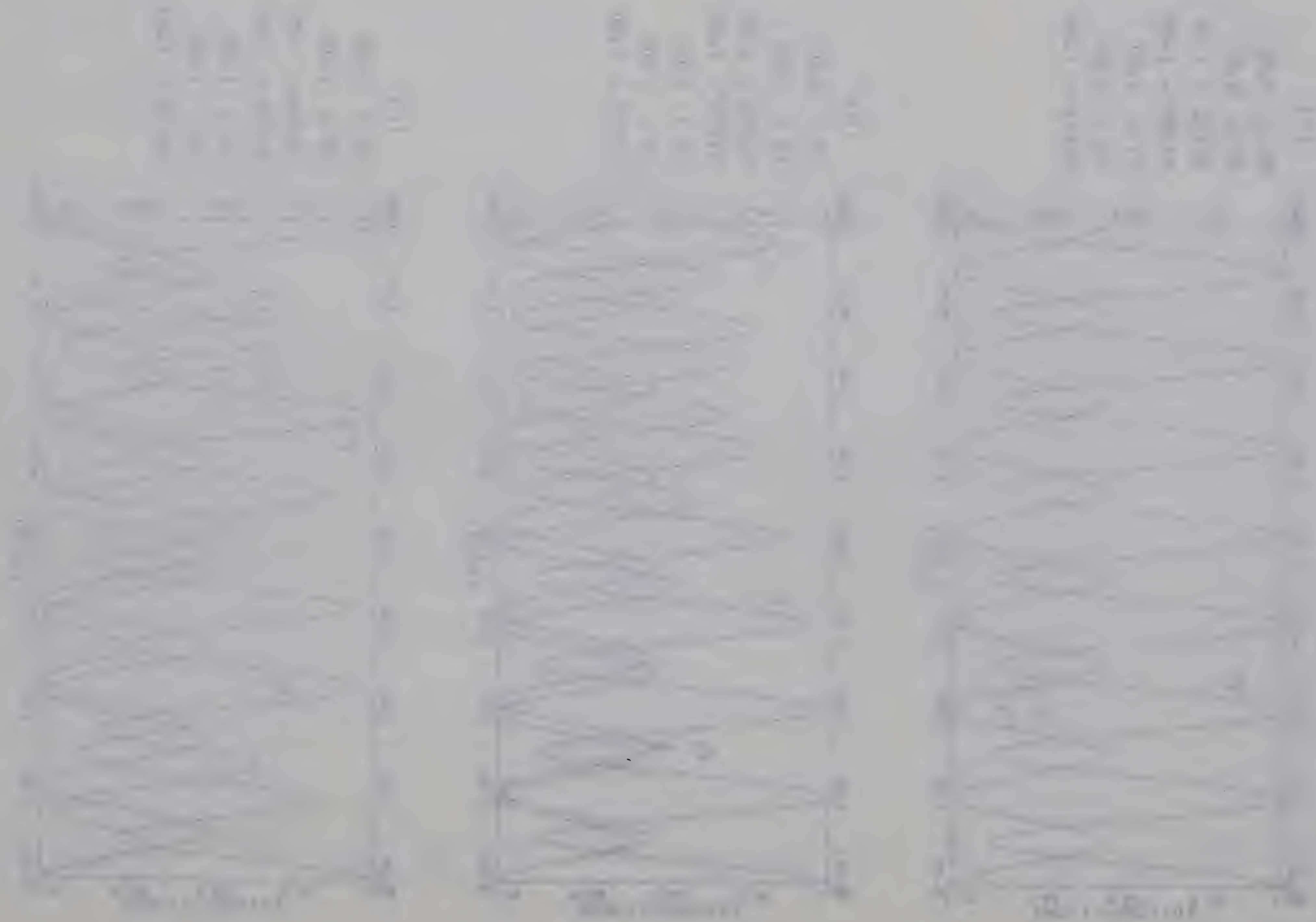


Fig. 2.3. Equal mass systems with r_3 = 0, 0.2 and 1.



Case IV Long period oscillations

In the two-channel model the density oscillations are due to the factor

$$\cos[(\lambda^{(2)}-\lambda^{(1)})\frac{t}{\hbar}] = \cos[\sqrt{1+\frac{1}{4}(x^2-r)^2}\ \frac{2V^{(12)}}{\hbar}\ t]. \quad (2.40)$$

Thus the maximum period possible is $T = h/2V^{(12)}$ occurring when $x^2 = r$. For example if $V^{(12)} \sim 10$ eV then $T \sim 10^{-15}$ seconds. However in an n-channel model, as n increases the periods can become much longer due to beats or to closely lying eigenvalues. This is only possible for $n \geq 3$. The reason is that in a 2 channel system, $V^{(12)}$ causes the smaller eigenvalue to be less in value than the smaller diagonal element and the larger eigenvalue to be greater than the larger diagonal element. Due to this widening, the 2 channel system has a maximum period. This argument can be applied to the n channel system. By rotating two axes at a time (method of Jacobi rotation) until the hamiltonian is diagonalized, we see that the smallest diagonal element must decrease, and the largest must increase. But it is possible in the 3 channel system, for the intermediate eigenvalue to lie close to one of the other two, causing a long period oscillation.

An example is the case where $r_2 = 2$, $r_3 = 4$, $m_1 = m_2 = m_3$, $U_1 = 1$, $U_2 = 3$. Then

$$\frac{1}{V^{(12)}}\underset{\approx}{\varepsilon} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{pmatrix} + x^2 I = M + x^2 I \ . \qquad (2.41)$$

Solving $|\mu I - M| = 0$ we get $\mu_1 = 0$, $\mu_{2,3} = 3 \pm \sqrt{12}$. Thus the three periods are $T_1 \sim .62\,\pi$, $T_2 \sim 8.62\pi$ and $T_3 \sim .58\,\pi$. It can be shown that the coefficient of the cos $\omega_2 \tau$ factor is relatively large. Thus the density oscillations consist of a large amplitude-long period oscillation with a small amplitude-short period oscillation superimposed on it, as shown in Fig. 2.4(b). Figs. 2.5 and 2.6 show systems with parameters slightly different from those of Fig. 2.4, and with the long period absent.

Systems with particles of unequal mass

If the masses of the a, b, and c particles are not all equal, the time evolution equations are much more complicated than those for the preceding examples. The unitary matrix U and the eigenvalues P_i are functions of the integration variable x, so we must use the general equation (2.30) rather than eqn. (2.34).

It is still possible to make approximations to (2.30) for small times and theoretically at least, for large times. For $\tau \sim 0$, the arguments of the cosines are small and the cosine factors oscillate

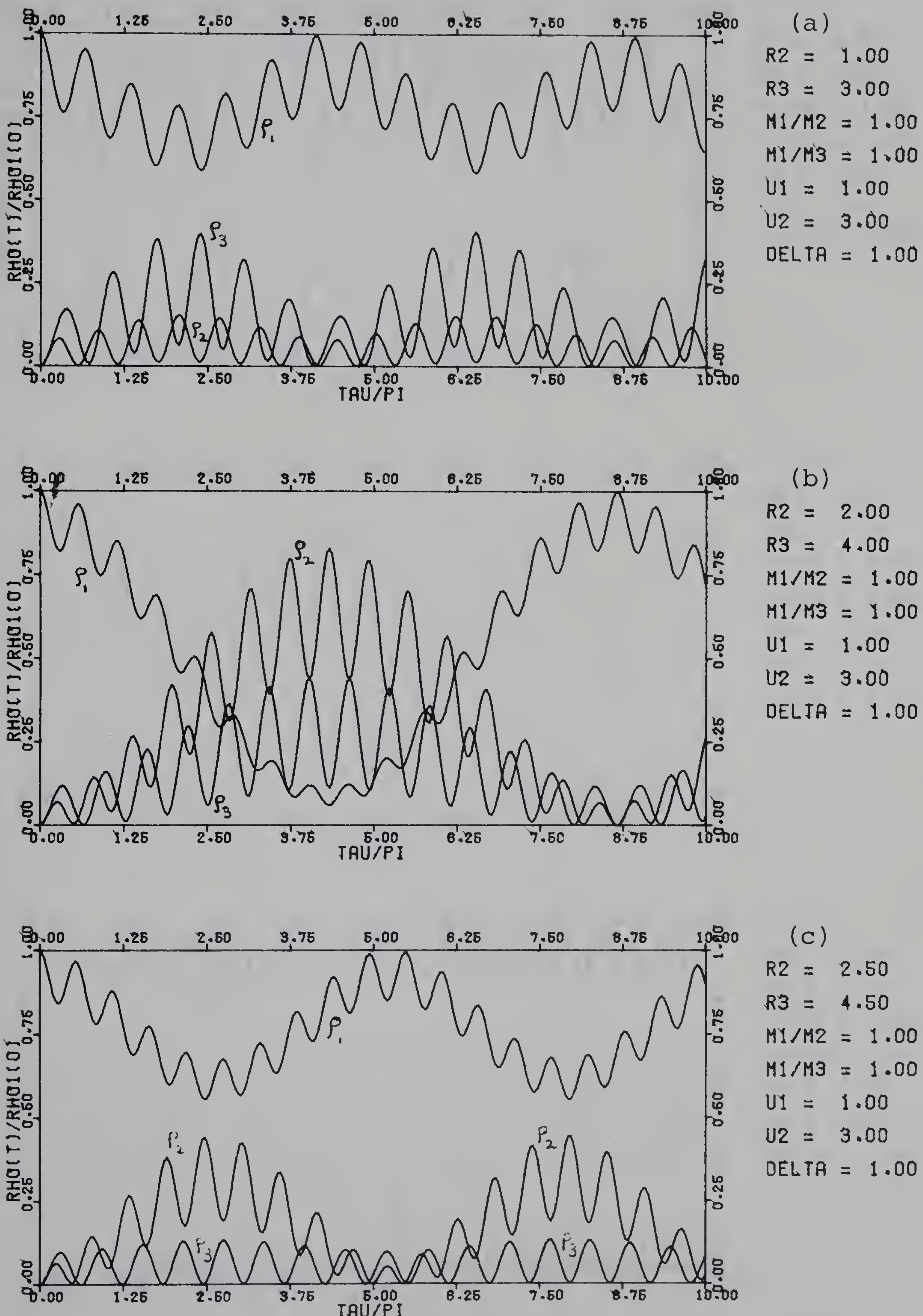


Fig. 2.4. Long period oscillations in the 3-channel system.

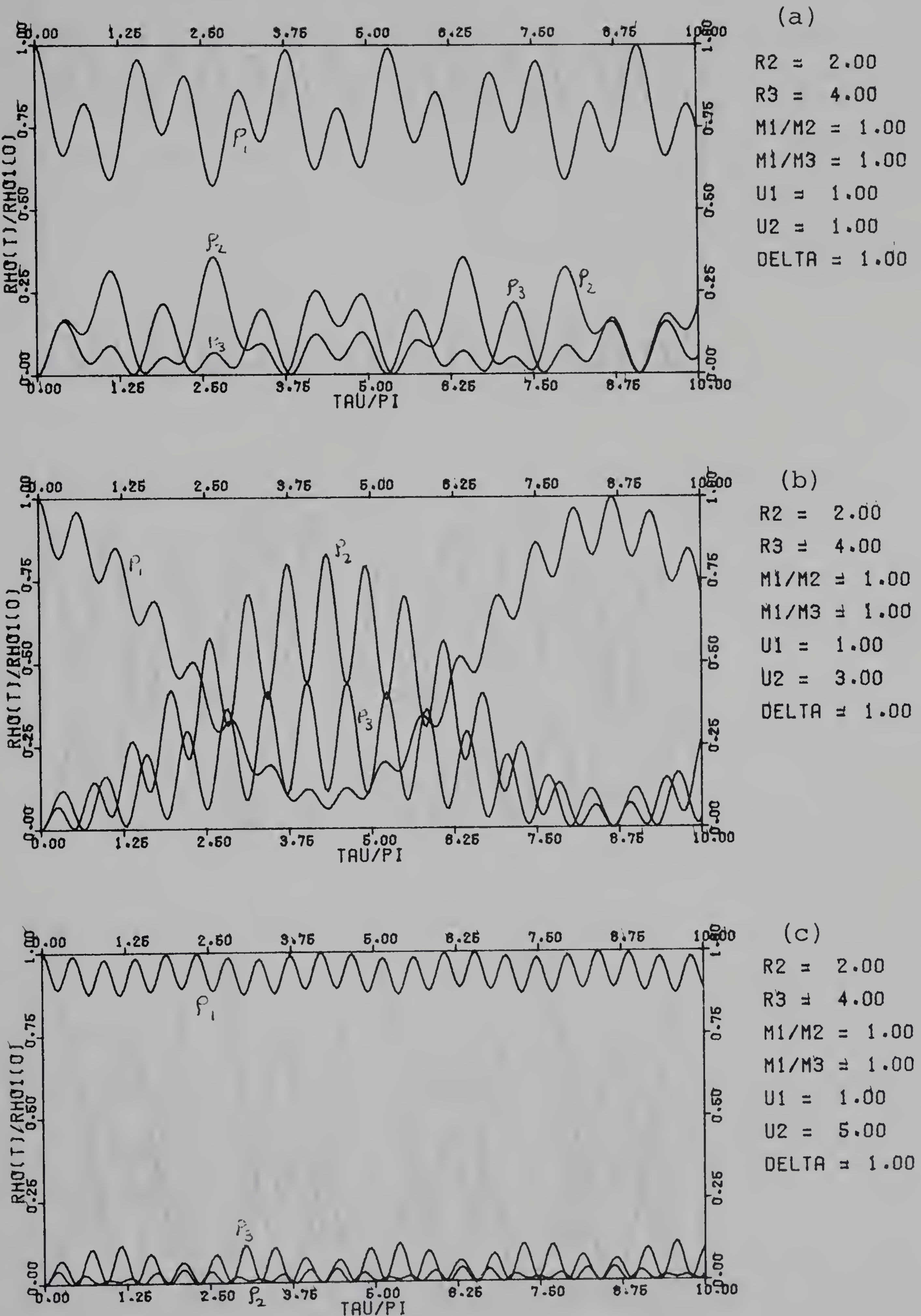

(a)
R2 = 2.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00
RHO(T)/RHO1(0)
TAU/PI
ρ_1
ρ_2
ρ_3
(b)
R2 = 2.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 3.00
DELTA = 1.00
RHO(T)/RHO1(0)
TAU/PI
ρ_1
ρ_2
ρ_3
(c)
R2 = 2.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 5.00
DELTA = 1.00
RHO(T)/RHO1(0)
TAU/PI
ρ_1
ρ_2
ρ_3

Fig. 2.5.

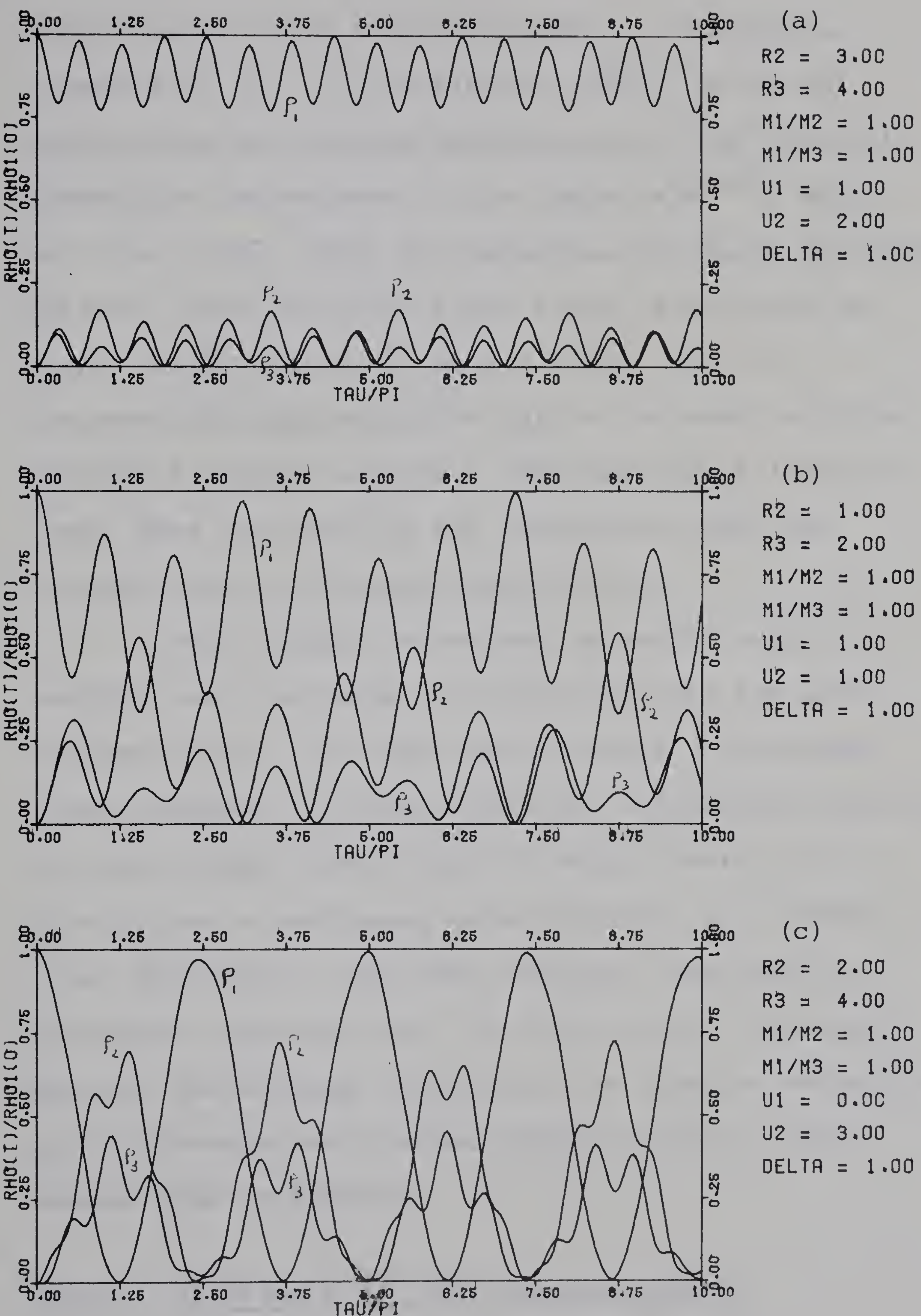

(a)
R2 = 3.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 2.00
DELTA = 1.00
(b)
R2 = 1.00
R3 = 2.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 1.00
U2 = 1.00
DELTA = 1.00
(c)
R2 = 2.00
R3 = 4.00
M1/M2 = 1.00
M1/M3 = 1.00
U1 = 0.00
U2 = 3.00
DELTA = 1.00
RHO(T)/RHO1(0)
TAU/PI
0.00
1.25
2.50
3.75
5.00
6.25
7.50
8.75
10.00
0.25
0.50
0.75
1.00
ρ_1
ρ_2
ρ_3

Fig. 2.6.

slowly since the P_i's are of order 1. The matrix elements A, B, ...J are also of order 1 so we can assume that the maximum contribution to the integral comes from the maximum of the factor $x^2e^{-\delta x^2}$, which is at $x = 1/\sqrt{\delta}$. Thus the densities are again described by eqn. (2.34) but with U and the P_i's evaluated by eqns. (2.32) and (2.33) at $x = 1/\sqrt{\delta}$. Fig. 2.7 compares this approximation (a) to the exact solution (b) for a typical example. Note that now δ (temperature) does not drop out but determines where the integral has its maximum contribution.

For τ large the cosines generally oscillate rapidly over the range of integration and the above argument fails. In this domain Kelvin's Stationary Phase Argument [3] states that the maximum contribution to the integral comes from the region where $\omega_i(x) = P_j - P_k/2$ has a stationary point giving a $1/\sqrt{\tau}$ damped time dependence. The other regions' contributions interfere destructively. If there are no stationary points, the maximum contribution is from the endpoints of the integration interval giving a $1/\tau$ or faster damped time dependence.

Case V Altering $V^{(23)}$ with unequal masses

A beautiful example of this small and large time evolution is a system very similar to that shown

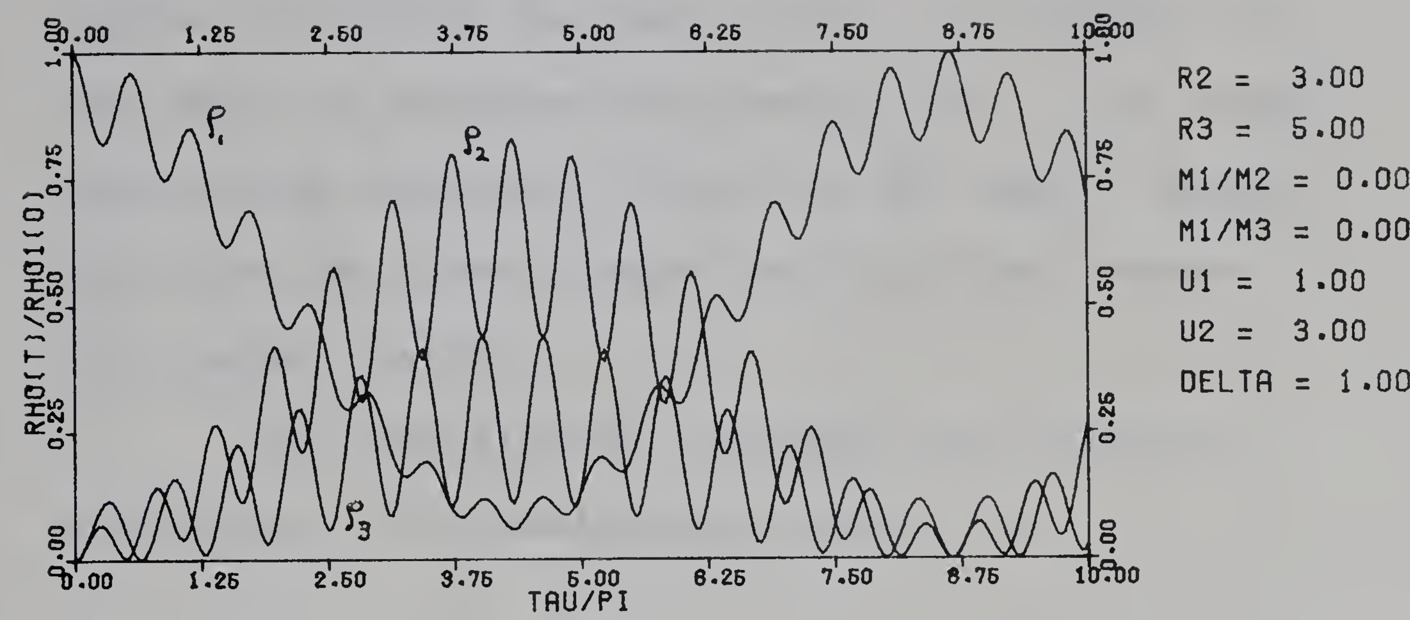


Fig. 2.7(a) Approximate solution (evaluating U and P_i, $i = 1,2,3$ at the point of maximum contribution $x = \delta^{-\frac{1}{2}}$).

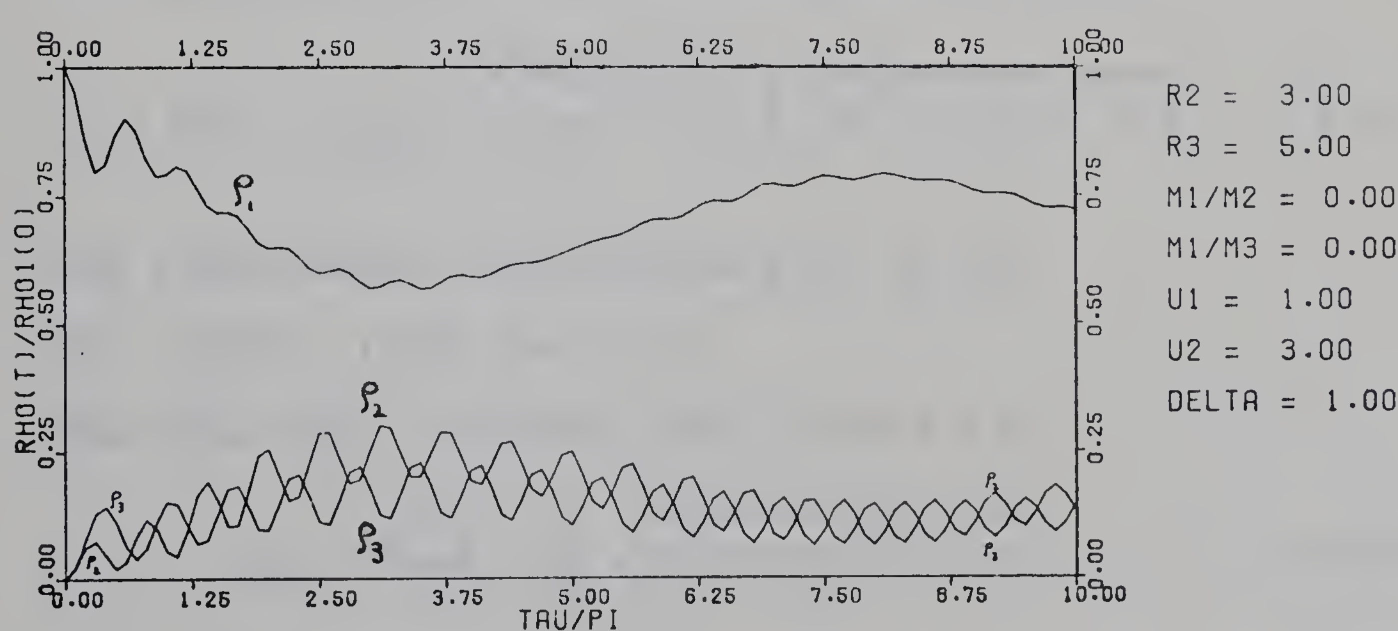
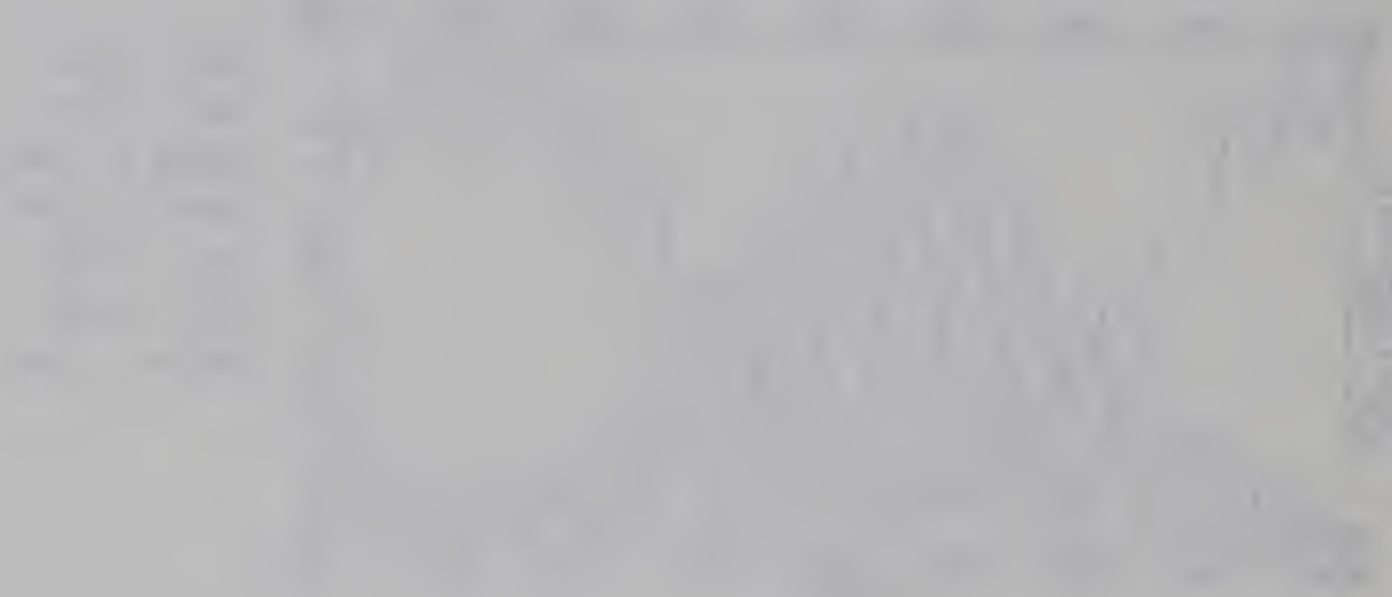
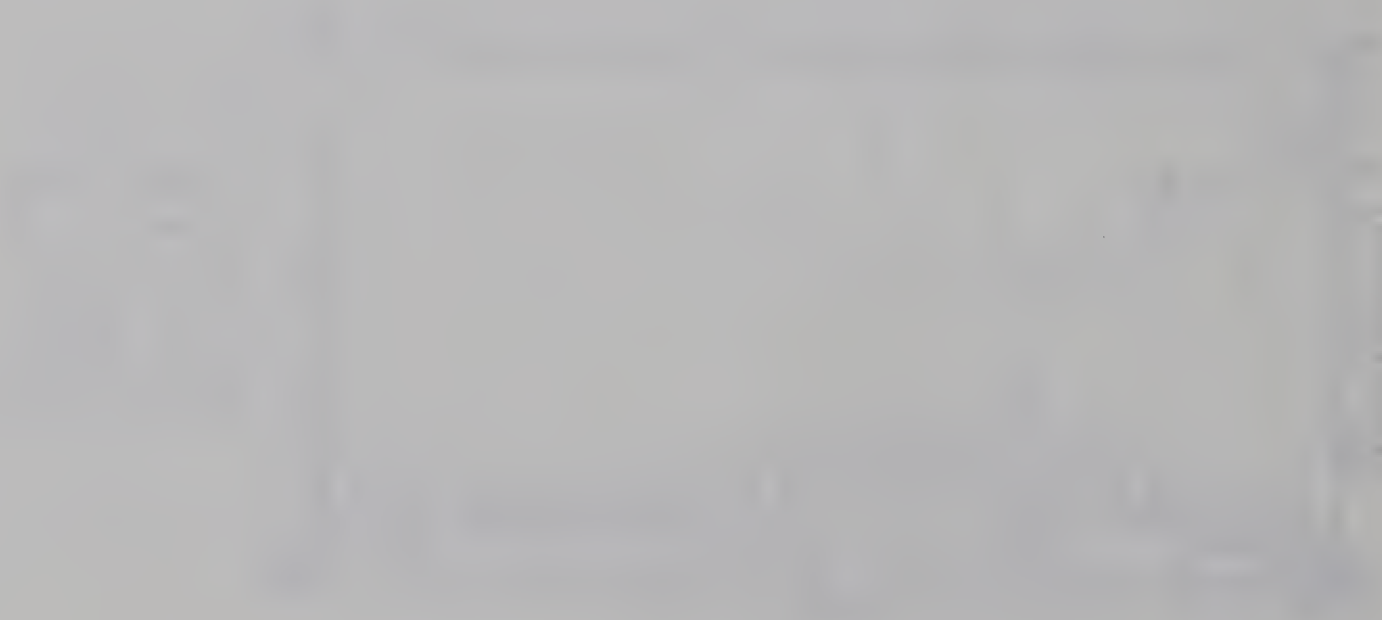


Fig. 2.7(b) Exact solution.





in Fig. 2.1. This time we let $m_1/m_2 = m_1/m_3 = 0$, $r_2 = r_3 = \delta = U_1 = 1$ and we again vary U_2. This system is similar to that in Fig. 2.1 because at the point of maximum contribution ($x = 1/\sqrt{\delta}$) their hamiltonian matrices (2.36) are the same. However this time we expect damped oscillations because of the unequal masses.

The oscillation frequency can be found explicitly. The hamiltonian matrix is

$$\frac{1}{V^{(12)}} \underset{\approx}{\varepsilon} = \begin{pmatrix} x^2 & 1 & U_1 \\ 1 & r_2 & U_2 \\ U_1 & U_2 & r_3 \end{pmatrix} = \begin{pmatrix} x^2-1 & 1 & 1 \\ 1 & 0 & U_2 \\ 1 & U_2 & 0 \end{pmatrix} + I = M+I \quad . \tag{2.42}$$

The solutions of the equation $|\lambda I - M| = 0$ are

$$\lambda_1 = -U_2, \quad \lambda_{2,3} = \frac{x^2-(1-U_2)}{2} \pm \frac{1}{2}\sqrt{(x^2-(1+U_2))^2+8} \quad . \tag{2.43}$$

The eigenvector corresponding to λ_1 is $(0, +1/\sqrt{2}, -1/\sqrt{2})$ so $A = 0$.

Thus the only frequency that enters is

$$\omega_3 = \frac{\lambda_3-\lambda_2}{2} = \frac{1}{2}\sqrt{(x^2-(1+U_2))^2+8}. \quad . \tag{2.44}$$

ω_3 has a stationary point at $x^2 = 1 + U_2$, so the period for asymptotically large times is $T = 2\pi/\omega = \sqrt{2}\,\pi$. For small times the frequency is

$$\omega_3\Big|_{x=1/\sqrt{\delta}} = \frac{1}{2}\sqrt{U_2^2 + 8}$$

(for $\delta = 1$) and the period is $T = 4\pi/\sqrt{U_2^2 + 8}$. Thus we expect initially a period dependent on U_2 (longer for small U_2, and shorter for large U_2) and finally for large τ a period of $\sqrt{2}\,\pi$ independent of U_2. Furthermore the oscillations should be damped as $1/\sqrt{\tau}$. This behaviour is illustrated in Fig. 2.8. In 2.8(a), $T = \sqrt{2}\,\pi$ throughout. In 2.8(b) the period increases from $T = \frac{4}{3}\pi$ to $T = \sqrt{2}\,\pi$. And in 2.8(c) the period dramatically increases from $T = 4\pi/\sqrt{24} \approx .8\,\pi$ to $T = \sqrt{2}\,\pi$ at about time $\tau = 2\pi$ or 3π.

Case VI Altering the c-particle mass, m_3

As a final example we study the trend in the density oscillations as the c particle mass changes. We choose the parameters $U_1 = U_2 = r_2 = r_3 = \delta = 1$ and $m_1/m_2 = 0$. Since eqn. (2.30) in this case cannot be solved analytically, we again make the approximation that the maximum contribution to the integral in (2.30) arises from the point $x = \delta^{-1/2}$. Then the hamiltonian equation is

$$\frac{1}{V^{(12)}}\,\underset{\approx}{\varepsilon} = M + I \qquad (2.45)$$

where

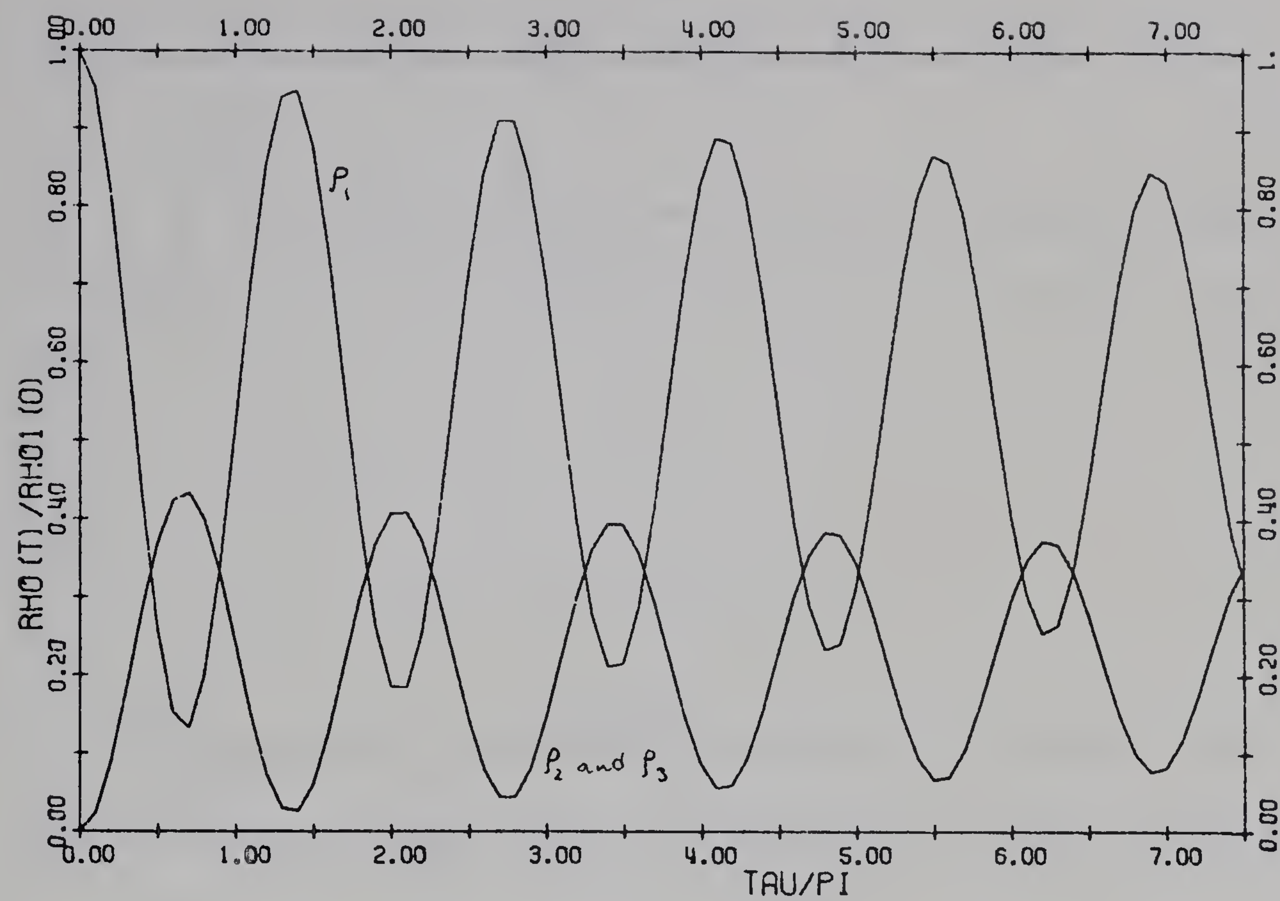


Fig. 2.8(a) $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $r_2 = r_3 = \delta = U_1 = 1.$; $U_2 = 0.$

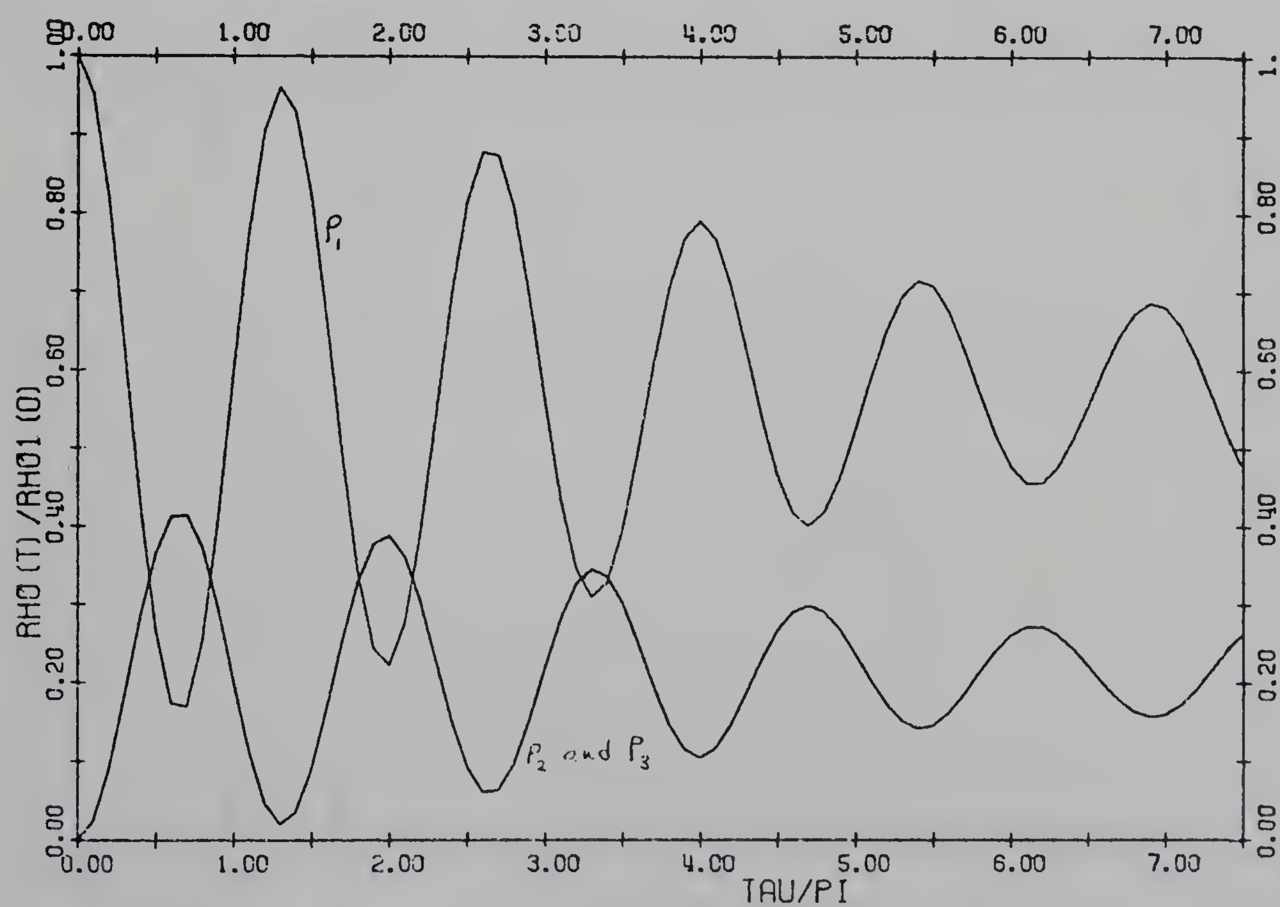


Fig. 2.8(b) $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $r_2 = r_3 = \delta = U_1 = 1.$; $U_2 = 1.$

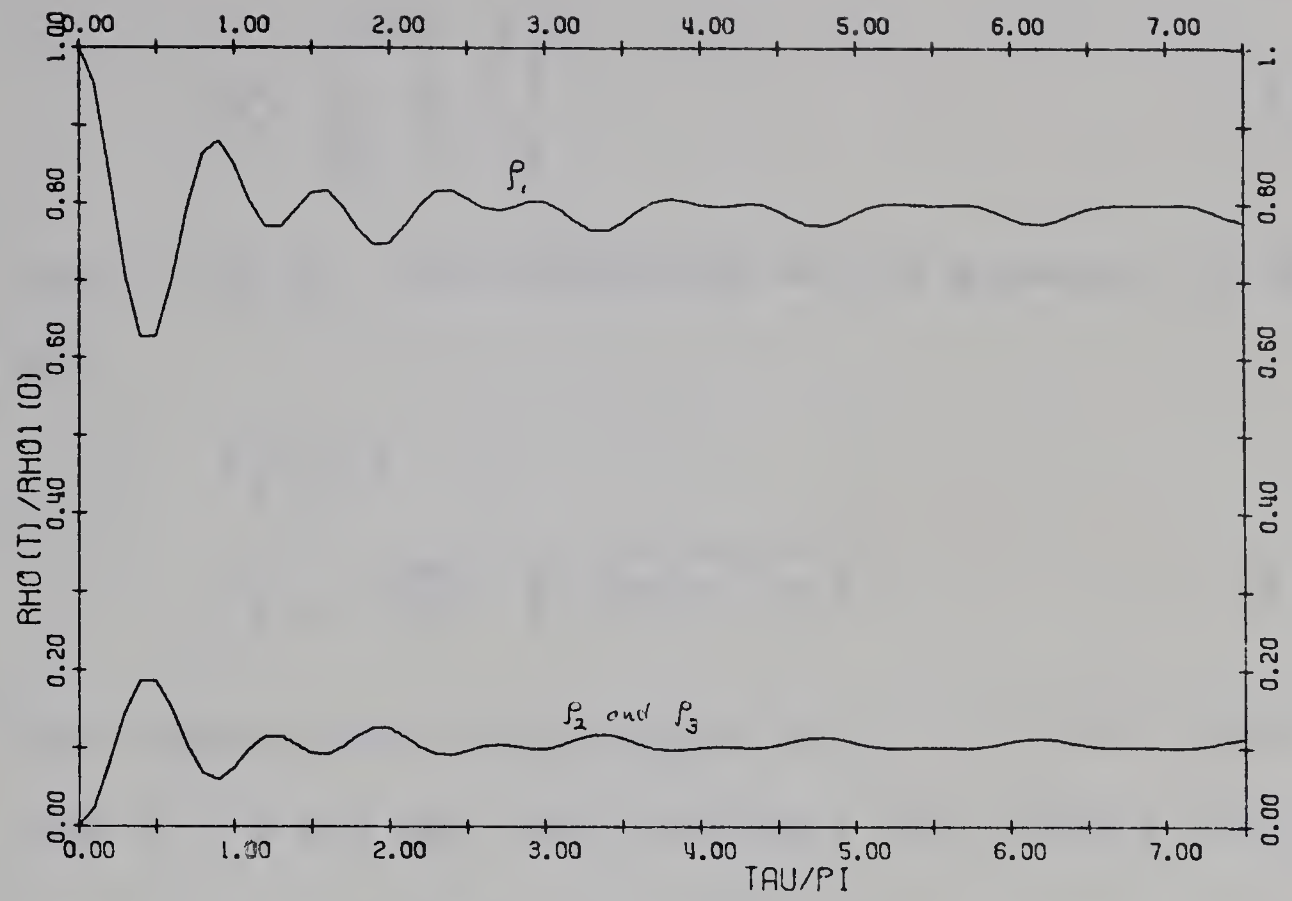


Fig. 2.8(c) $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.;\ r_2 = r_3 = \delta = U_1 = 1.;\ U_2 = 4.$

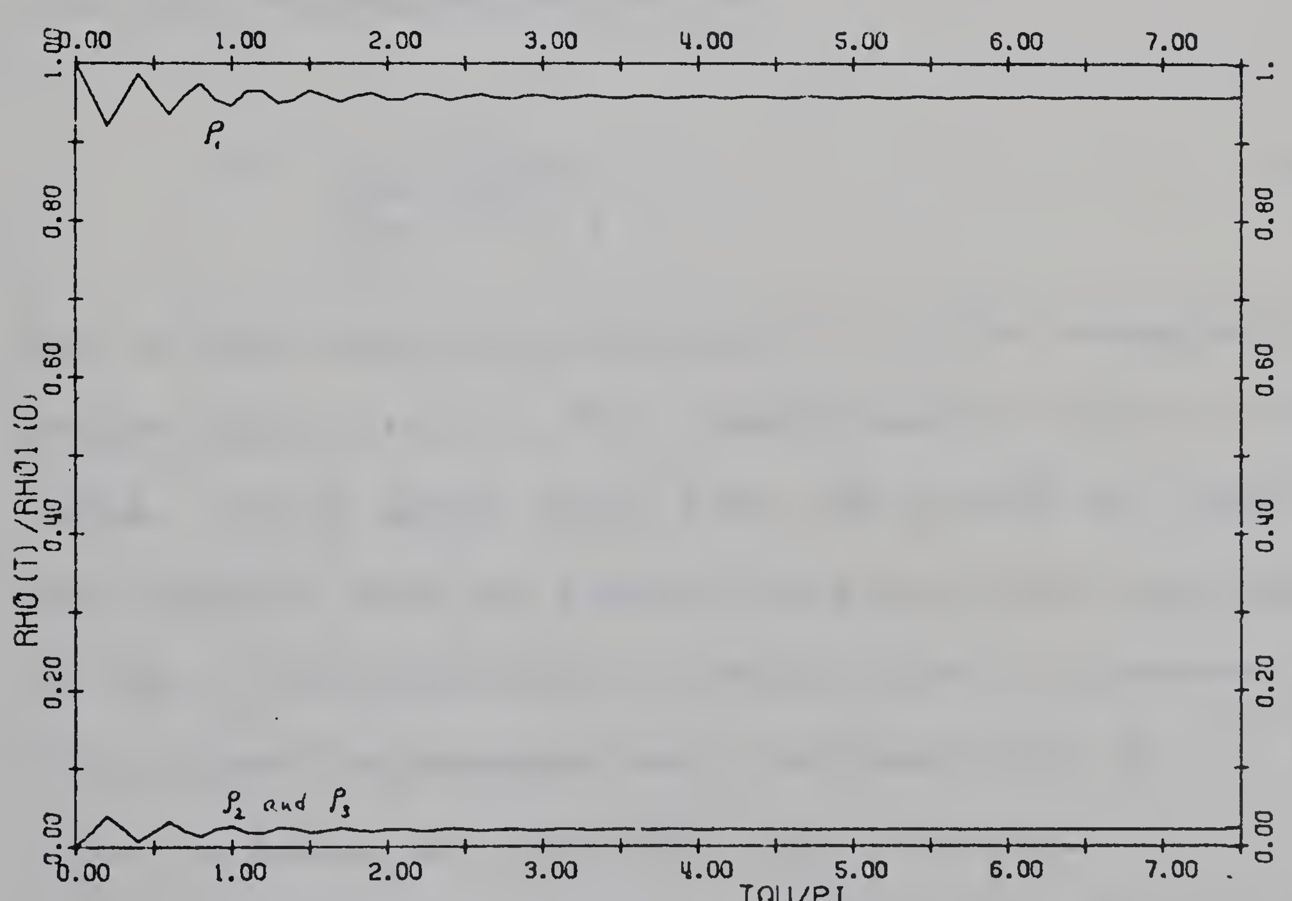


Fig. 2.8(d) $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.;\ r_2 = r_3 = \delta = U_1 = 1.;\ U_2 = 10.$

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & m \end{pmatrix} \tag{2.46}$$

and $m = m_1/m_3$. The solutions of the equation $|\mu I - M| = 0$ are

$$\mu_1 = -1$$

$$\mu_{2,3} = \frac{m+1}{2} \pm \frac{1}{2}\sqrt{(m-1)^2 + 8} \quad . \tag{2.47}$$

The eigenvector corresponding to μ_1 is $(1/\sqrt{2}, -1/\sqrt{2}, 0)$. Thus $C = 0$ and the only frequency that enters for ρ_3 is

$$\omega_3 = \frac{\mu_2 - \mu_3}{2} = \frac{1}{2}\sqrt{(m-1)^2 + 8} \tag{2.48}$$

with the corresponding period

$$T = \frac{4\pi}{\sqrt{(m-1)^2 + 8}} \quad . \tag{2.49}$$

Due to the stationary point at $m = 1$ the asymptotic period for ρ_3 is $T = \sqrt{2}\,\pi$ regardless of initial conditions. If we again guess that the period and amplitude are coupled, then we expect the period and amplitude of the ρ_3 oscillations to increase as m increases from 0 to 1, and to decrease as m increases for $m > 1$. This trend is borne out in Figs. 2.9(a) to (d).

Another interesting point is that in the limit of large m the eigenvalues are $\mu_1 = -1$, $\mu_2 = 1$, and $\mu_3 = m$. The eigenvector corresponding to μ_3 in this

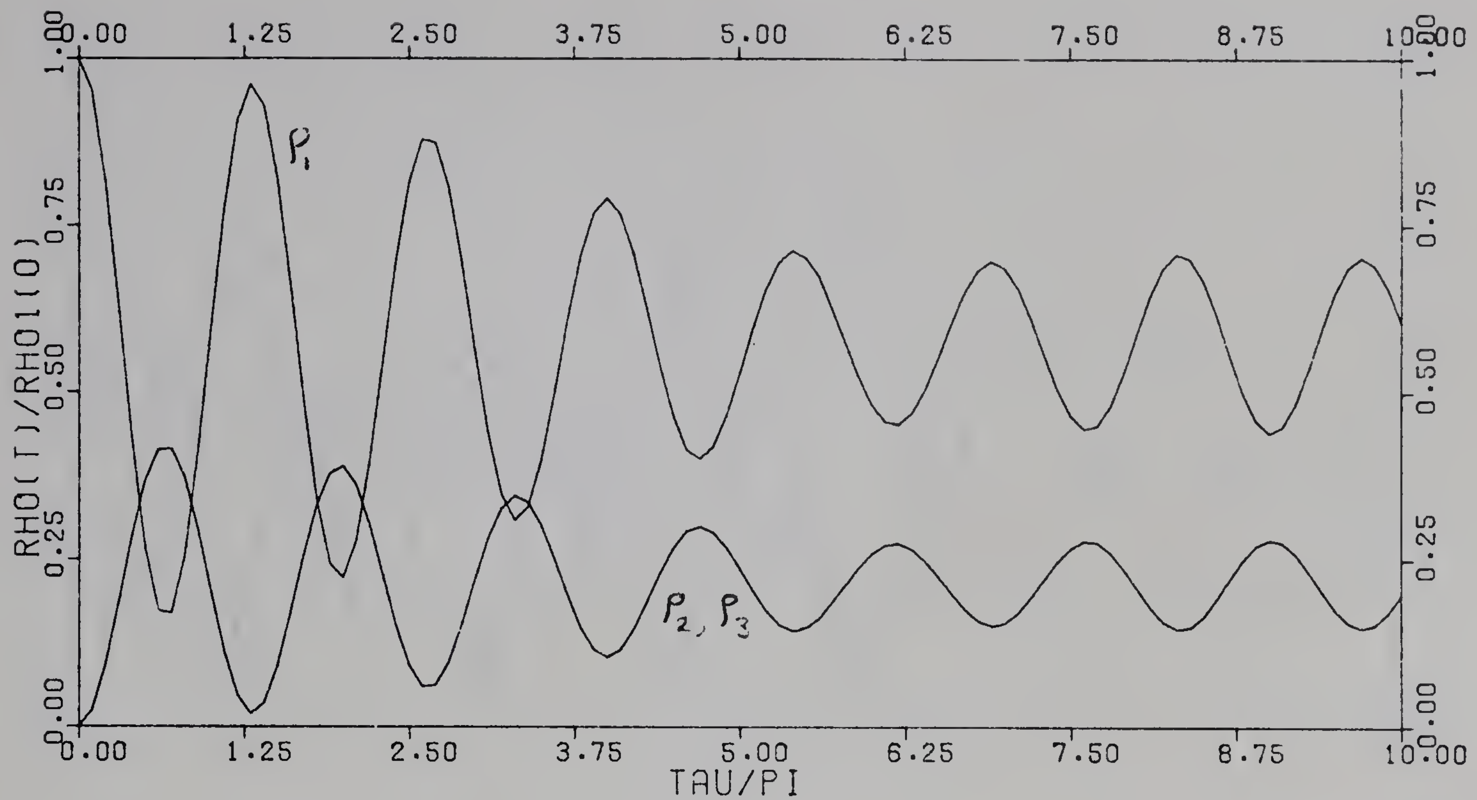


Fig. 2.9(a) $r_2 = r_3 = 1.;\ \frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.;\ U_1 = U_2 = \delta = 1.$

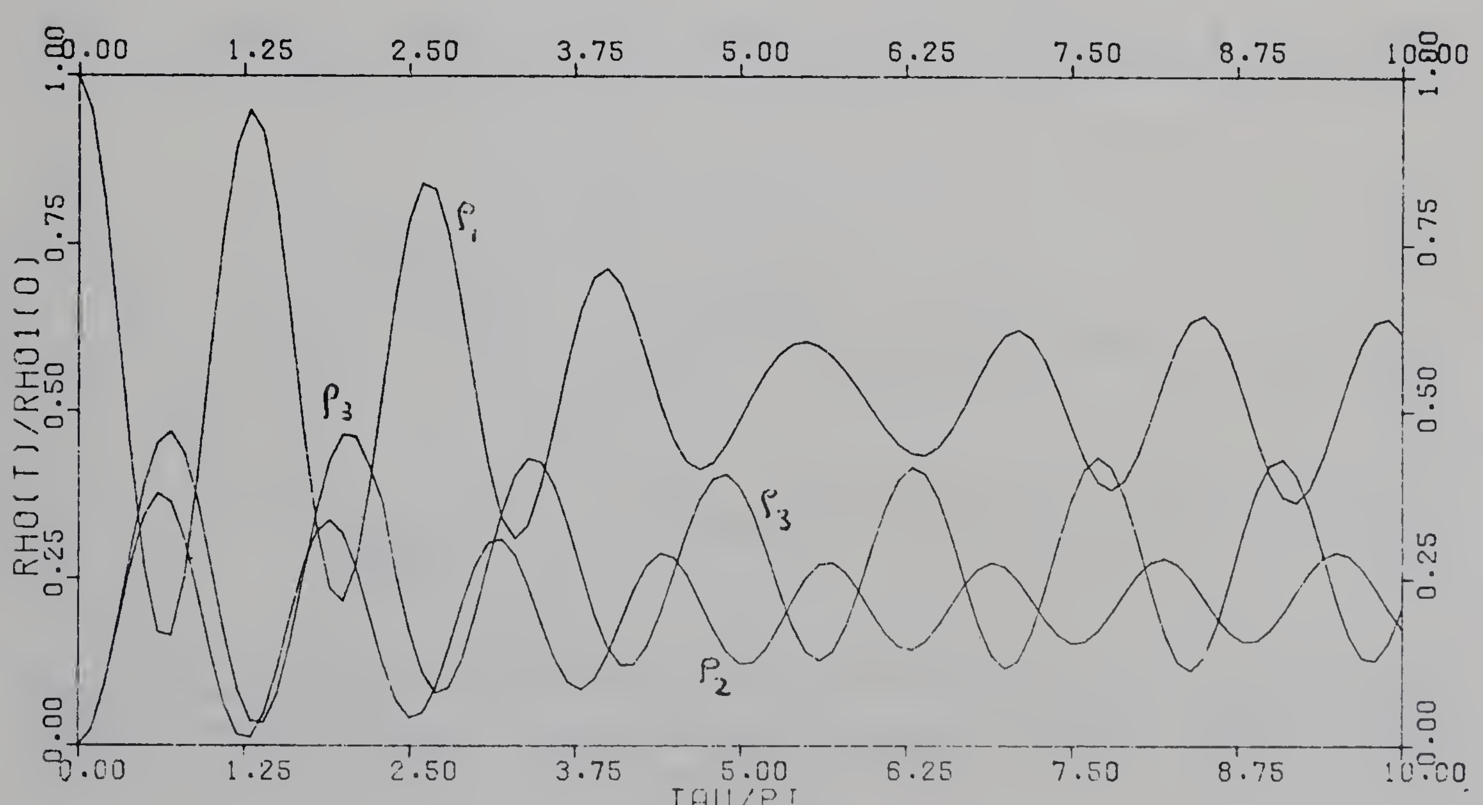


Fig. 2.9(b) $r_2 = r_3 = 1.;\ \frac{m_1}{m_2} = 0.;\ \frac{m_1}{m_3} = 0.2.;\ U_1 = U_2 = \delta = 1.$

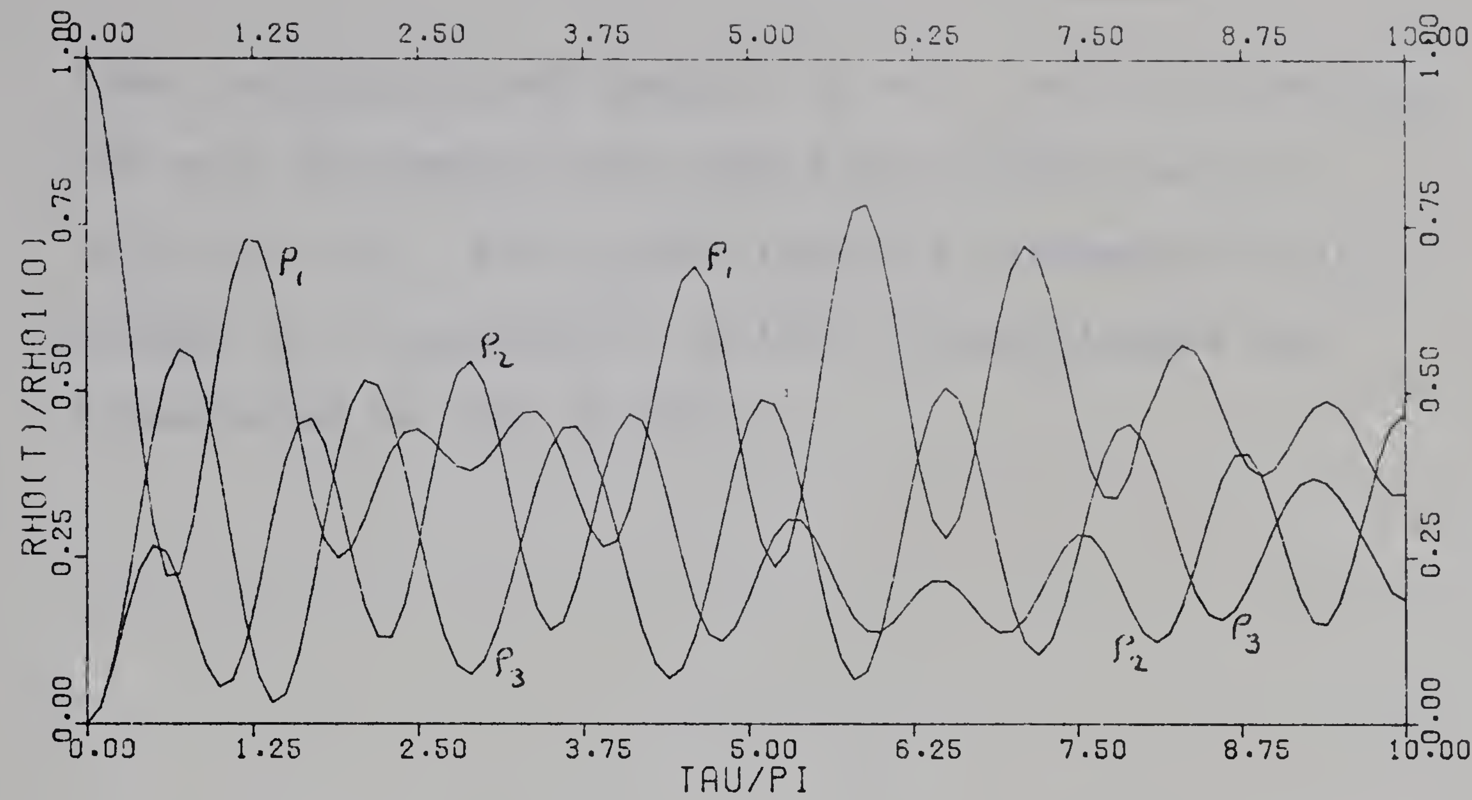


Fig. 2.9(c) $r_2 = r_3 = 1.;\ \frac{m_1}{m_2} = 0.;\ \frac{m_1}{m_3} = 1.;\ U_1 = U_2 = \delta = 1.$

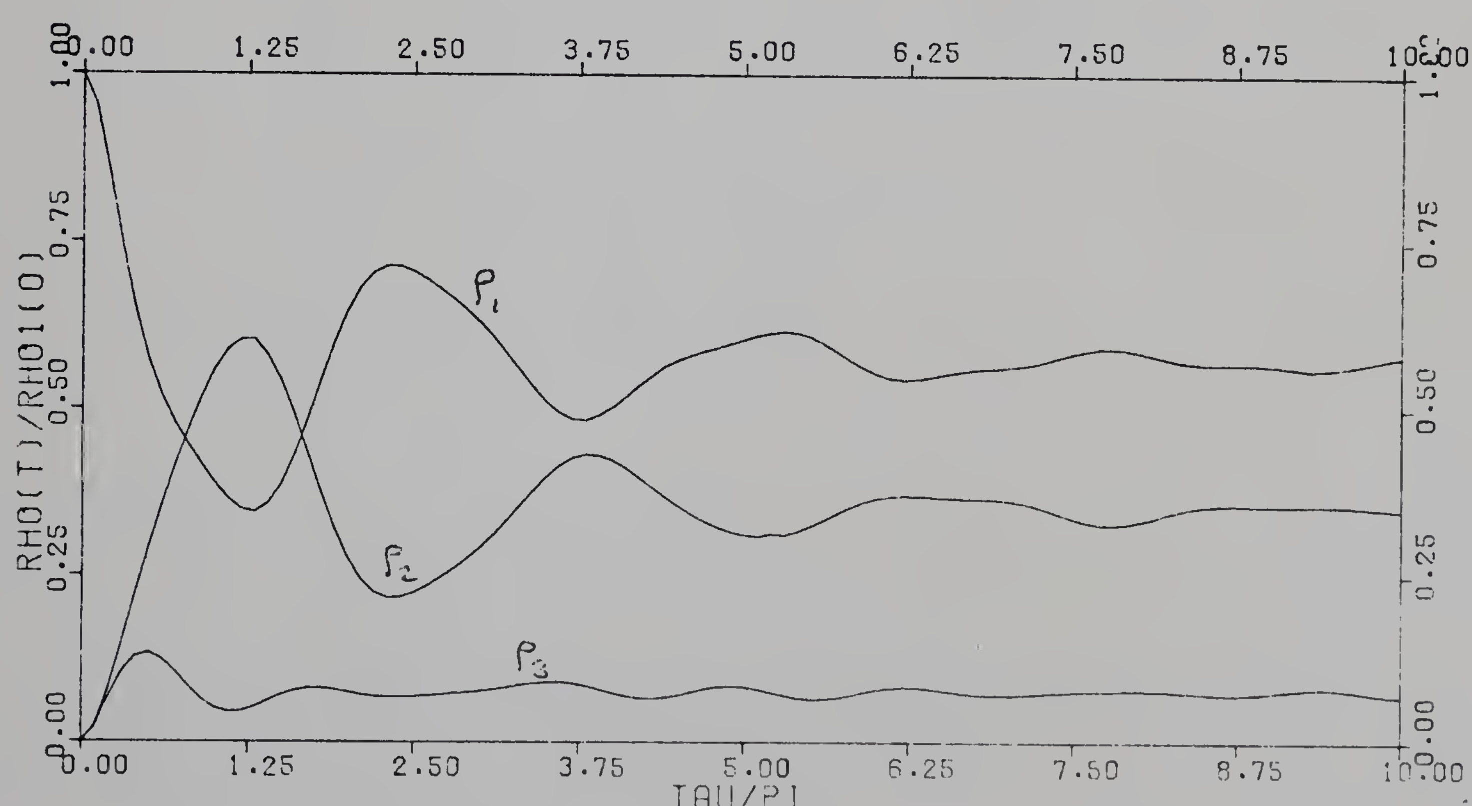


Fig. 2.9(d) $r_2 = r_3 = 1.;\ \frac{m_1}{m_2} = 0.;\ \frac{m_1}{m_3} = 5.;\ U_1 = U_2 = \delta = 1.$

limit is (0,0,1) and thus G = H = 0. So for ρ_1 and ρ_2 the only frequency that enters for large m is $\omega_1 = (\mu_2-\mu_1)/2 = 1$. For ρ_3 the limiting frequency is of course $\omega_3 = (\mu_3-\mu_2)/2 = (m-1)/2$. These trends are illustrated in Fig. 2.9(d).

§2.3 The n-Channel System

We consider n kinds of particles $a^{(1)}, a^{(2)}, \ldots a^{(n)}$ (all bosons or all fermions) which can be transformed into each other via external fields.

The hamiltonian for this sytem is

$$\mathcal{H} = \sum_{i=1}^{n} \sum_{k} \varepsilon_k^{(i)} a_k^{(i)\dagger} a_k^{(i)} + \theta(t) \sum_{i<j} \sum_{kk'} V_{kk'}^{(ij)} (a_k^{(i)\dagger} a_{k'}^{(j)} + + a_k^{(j)\dagger} a_{k'}^{(i)}) \quad . \tag{2.50}$$

If we assume that the external fields $V^{(ij)}$ are constant, then $V_{kk'}^{(ij)} = V^{(ij)} \delta_{kk'}$ for $i,j = 1,2,\ldots n$; $i < j$. Then

$$\mathcal{H} = \sum_{k} \{ \sum_{i=1}^{n} \varepsilon_k^{(i)} a_k^{(i)\dagger} a_k^{(i)} + \theta(t) \sum_{i<j} V^{(ij)} (a_k^{(i)\dagger} a_k^{(j)} + + a_k^{(j)\dagger} a_k^{(i)}) \}$$

$$= \sum_{k} \{ \sum_{i,j=1}^{n} a_k^{\dagger(i)} (E_k)_{ij} a_k^{(j)} \} \tag{2.51}$$

where

$$(E_k)_{ij} = \varepsilon_k^{(i)} \delta_{ij} + (1 - \delta_{ij}) V^{(ij)} \quad . \tag{2.52}$$

This hamiltonian equation can be diagonalized with a unitary matrix, U. That is

$$\mathcal{H} = \sum_{k} \{ \sum_{i} \alpha_k^{(i)\dagger} \alpha_k^{(i)} \lambda_k^{(i)} \} \tag{2.53}$$

where

$$\lambda_k^{(i)} = \sum_{j\ell} (U_k)_{ij} (E_k)_{j\ell} (U_k^*)_{i\ell} \tag{2.54}$$

and where we have introduced quasiparticles by a linear transformation

$$\alpha_k^{(i)} = \sum_j (U_k)_{ij} a_k^{(j)} \tag{2.55}$$

with inverse transformation

$$a_k^{(i)} = \sum_\ell (U_k^*)_{\ell i} \alpha_k^{(\ell)} \quad . \tag{2.56}$$

The quasiparticles obey the same commutation relations as the particles. That is, with no momentum mixing, if

$$[a_k^{(i)}, a_k^{(j)\dagger}]_\pm = \delta_{ij} \qquad \frac{F-D}{B-E} \tag{2.57}$$

$$[a_k^{(i)}, a_k^{(j)}]_\pm = 0$$

then

$$[\alpha_k^{(i)}, \alpha_k^{(j)\dagger}]_\pm = \delta_{ij} \qquad \frac{F-D}{B-E} \tag{2.58}$$

$$[\alpha_k^{(i)}, \alpha_k^{(j)}]_\pm = 0 \quad .$$

For example,

$$\begin{aligned}[\alpha_k^{(i)}, \alpha_k^{(j)\dagger}] &= \sum_{\ell m} (U_k)_{i\ell} a_k^{(\ell)} (U_k^*)_{jm} a_k^{\dagger(m)} \\ &\pm \sum_{\ell m} (U_k^*)_{jm} a_k^{\dagger(m)} (U_k)_{i\ell} a_k^{(\ell)}\end{aligned}$$

$$= \sum_{\ell m} (U_k)_{i\ell} (U_k^*)_{jm} [a_k^{(\ell)}, a_k^{\dagger(m)}]_\pm$$

$$= \sum_{\ell} (U_k)_{i\ell} (U_k^*)_{j\ell}$$

$$= \delta_{ij} \quad .$$

Theorem One and its corollary from Chapter One are unchanged for n particles. That is

$$\alpha_k^{(i)} \mathcal{H} = \mathcal{H} \alpha_k^{(i)} + \lambda_k^{(i)} \alpha_k^{(i)} \tag{2.59}$$

$$\alpha_k^{(i)} e^{-i\mathcal{H}t/\hbar} = e^{-i\mathcal{H}t/\hbar} \alpha_k^{(i)} e^{-i\lambda_k^{(i)} t/\hbar} \quad . \tag{2.60}$$

Thus

$$e^{i\mathcal{H}t/\hbar} a_k^{(i)} e^{-i\mathcal{H}t/\hbar} = \sum_j (U_k^*)_{ji} e^{-i\lambda_k^{(j)} t/\hbar} \alpha_k^{(j)}$$

$$= \sum_j (U_k^*)_{ji} e^{-i\lambda_k^{(j)} t/\hbar} \sum_\ell (U_k)_{j\ell} a_k^{(\ell)}$$

$$= \sum_\ell \{ \sum_j (U_k^*)_{ji} (U_k)_{j\ell} e^{-i\lambda_k^{(j)} t/\hbar} \} a_k^{(\ell)}$$

$$= \sum_\ell \{ P_k^{i\ell} \} a_k^{(\ell)} \quad . \tag{2.61}$$

This enables us to calculate the time dependence of the number density of particles $a^{(i)}$ in state k:

$$n_k^{(i)}(t) = \frac{\mathrm{Tr}(a_k^{(i)\dagger} a_k^{(i)} \rho_t)}{\mathrm{Tr}(\rho_t)} \quad .$$

$$= \frac{\mathrm{Tr}\{ \sum_{\ell m} P_k^{*i\ell} a_k^{(\ell)\dagger} P_k^{im} a_k^{(m)} e^{-\beta \mathcal{H}_o} \}}{\mathrm{Tr}(e^{-\beta \mathcal{H}_o})} \tag{2.62}$$

where $e^{-\beta\mathcal{H}_o}$ is the equilibrium statistical operator for time $t \leq 0$. Using the fact that

$$\frac{Tr(a_k^{\dagger(\ell)} a_k^{(m)} e^{-\beta\mathcal{H}_o})}{Tr(e^{-\beta\mathcal{H}_o})} = \delta_{\ell m} \frac{1}{e^{-\beta\mu} e^{\beta\varepsilon_k^{(\ell)}} \pm 1}$$

$$= \delta_{\ell m} n^{(o)}(\varepsilon_k^{(\ell)}) \qquad \frac{F-D}{B-E} \qquad (2.63)$$

eqn. (2.62) becomes

$$n_k^{(i)}(t) = \sum_{\ell m} P_k^{*i\ell} P_k^{im} \delta_{\ell m} n^{(o)}(\varepsilon_k^{(\ell)})$$

$$= \sum_{\ell} |P_k^{i\ell}|^2 n^{(o)}(\varepsilon_k^{(\ell)}) \qquad (2.64)$$

where

$$|P_k^{i\ell}|^2 = |\sum_{jm} (U_k)_{mi} (U_k^*)_{ji} (U_k^*)_{m\ell} (U_k)_{j\ell} e^{i(\lambda_k^{(m)} - \lambda_k^{(j)})t/\hbar}|$$

$$= \tilde{P}_{ik\ell j} + \sum_{j<m} P_{ik\ell jm} \cos(\lambda_k^{(m)} - \lambda_k^{(j)}) \frac{t}{\hbar} \quad . \qquad (2.65)$$

Finally, the time evolution equation for the i^{th} species is

$$\rho_i(t) = \frac{1}{(2\pi)^3} \int d^3k \; n_k^{(i)}(t)$$

$$= \frac{1}{(2\pi)^3} \int d^3k \sum_{\ell} |P_k^{i\ell}|^2 n^{(o)}(\varepsilon_k^{(\ell)}) \quad . \qquad (2.66)$$

If there are only $a^{(j)}$ particles present at time $t = 0$ then

$$\rho_i(t) = \frac{1}{(2\pi)^3} \int d^3k \, |P_k^{ij}|^2 \, n^{(o)}(\varepsilon_k^{(j)}) \ . \tag{2.67}$$

An example of this type of system is

$$\begin{tikzcd}
 & a^{(1)} \arrow[d, leftrightarrow] & \\
 & a^{(2)} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
a^{(3)} \arrow[rr, leftrightarrow] & & a^{(4)}
\end{tikzcd}$$

Here, $\underset{\approx}{\varepsilon}$ would be

$$\begin{pmatrix}
E^{(1)} & V^{(12)} & 0 & 0 \\
 & E^{(2)} & V^{(23)} & V^{(24)} \\
 & & E^{(3)} & V^{(34)} \\
 & & & E^{(4)}
\end{pmatrix}$$

CHAPTER III

ZERO TEMPERATURE-TWO CHANNEL FERMION SYSTEM

§3.1 Time Evolution Equation

In this chapter we derive the zero temperature limit for a two-channel system with both a and b particles present at time t = 0. We start with equation (1.21):

$$n_k^{(1)}(t) = n^{(o)}(\varepsilon_k^{(1)}) + \frac{1}{2}\frac{V_o^2}{V_o^2+\xi_k^2}\left[1-\cos(\lambda_k^{(1)}-\lambda_k^{(2)})\frac{t}{\hbar}\right] \times$$

$$\times\ \{n^{(o)}(\varepsilon_k^{(2)}) - n^{(o)}(\varepsilon_k^{(1)})\} \qquad (3.1)$$

where

$$n^{(o)}(\varepsilon_k^{(1)}) = [e^{-\beta\mu_1} e^{\beta\frac{\hbar^2k^2}{2m_1}} + 1]^{-1}$$

$$n^{(o)}(\varepsilon_k^{(2)}) = [e^{-\beta\mu_2} e^{\beta(\frac{\hbar^2k^2}{2m_2}+\varepsilon_o)} + 1]^{-1} . \qquad (3.2)$$

μ_1 and μ_2 are the Fermi energies of the a and b particles respectively. In the zero temperature limit the distribution functions (3.2) become

$$\lim_{\beta\to\infty} n^{(o)}(\varepsilon_k^{(1)}) = 1-\theta(\frac{\hbar^2k^2}{2m_1}-\mu_1) = 1-\theta(k-\frac{\sqrt{2m_1\mu_1}}{\hbar})$$

$$\lim_{\beta\to\infty} n^{(o)}(\varepsilon_k^{(2)}) = 1-\theta(\frac{\hbar^2k^2}{2m_2}+\varepsilon_o-\mu_2) = 1-\theta(k-\frac{\sqrt{2m_2(\mu_2-\varepsilon_o)}}{\hbar}) \qquad (3.3)$$

where

$$\theta(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} .$$

Before proceeding let us first calculate $\rho_1(0)$, $\rho_2(0)$ and the ratio $\rho_2(0)/\rho_1(0)$.

$$\rho_1(0) = \frac{1}{(2\pi)^3} \int d^3k \; n^{(o)}(\varepsilon_k^{(1)})$$

$$= \frac{2}{(2\pi)^2} \int_0^{\frac{\sqrt{2m_1\mu_1}}{\hbar}} k^2 dk = \frac{(2m_1\mu_1)^{3/2}}{6\pi^2\hbar^3}$$

$$\rho_2(0) = \frac{1}{(2\pi)^3} \int d^3k \; n^{(o)}(\varepsilon_k^{(2)})$$

$$= \frac{2}{(2\pi)^2} \int_0^{\sqrt{2m_2(\mu_2-\varepsilon_o)}/\hbar} k^2 dk = \frac{(2m_2(\mu_2-\varepsilon_o))^{3/2}}{6\pi^2\hbar^3} \tag{3.4}$$

$$\frac{\rho_2(0)}{\rho_1(0)} = \left[\frac{m_2}{m_1} \left(\frac{\mu_2 - \varepsilon_o}{\mu_1}\right)\right]^{3/2} .$$

Using the abbreviation

$$F(k,t) = \frac{1}{2} \frac{V_o^2}{V_o^2 + \xi_k^2} \left[1 - \cos(\lambda_k^{(1)} - \lambda_k^{(2)}) \frac{t}{\hbar}\right] \tag{3.5}$$

we get from eqn. (3.1) for the density of particles a in the thermodynamic limit

$$\rho_1(t) = \rho_1(0) + \int_0^{\sqrt{2m_2(\mu_2-\varepsilon_o)}/\hbar} 4\pi k^2 dk\ F(k,t)$$

$$- \int_0^{\sqrt{2m_1\mu_1}/\hbar} 4\pi k^2 dk\ F(k,t)$$

$$= \rho_1(0) - \int_{\frac{\sqrt{2m_2(\mu_2-\varepsilon_o)}}{\hbar}}^{\frac{\sqrt{2m_1\mu_1}}{\hbar}} 4\pi k^2 dk\ F(k,t)\ . \tag{3.6}$$

Defining the parameters x, r, τ and μ as in Chapter One, i.e.:

$$x = \frac{\hbar k}{\sqrt{2\mu V_o}} \qquad r = \frac{\varepsilon_o}{V_o} \qquad \tau = \frac{2V_o t}{\hbar} \qquad \mu = \frac{m_1 m_2}{m_2 - m_1} \tag{3.7}$$

the time evolution equation (3.6) for the a particle density can be written as

$$\rho_1(t) = \rho_1(0) \left\{ 1 - \frac{3}{2} \left(\frac{\mu}{m_1} \frac{V_o}{\mu_1}\right)^{3/2} \int_{\left(\frac{m_2}{\mu} \frac{\mu-\varepsilon_o}{V_o}\right)^{1/2}}^{\left(\frac{m_1}{\mu} \frac{\mu_1}{V_o}\right)^{1/2}} \times \right.$$

$$\left. \times \frac{x^2 dx}{1 + \frac{1}{4}(x^2 - r)^2} \left[1 - \cos \tau \sqrt{1 + \frac{1}{4}(x^2-r)^2}\right] \right\}$$

or finally as

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} = \frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} \left\{1 - \frac{3}{2}\,\delta^{3/2} \int_{\frac{1}{\sqrt{\delta}}\left(\frac{\rho_2(0)}{\rho_1(0)}\right)^{1/3}}^{1/\sqrt{\delta}} \right.$$

$$\left. \times \frac{x^2 dx}{1+\frac{1}{4}(x^2-r)^2}\left[1 - \cos\tau\sqrt{1+\frac{1}{4}(x^2-r)^2}\right]\right\} \qquad (3.8)$$

where

$$\delta = \frac{\mu}{m_1}\,\frac{V_o}{\mu_1} \quad .$$

§3.2 Kelvin's Stationary Phase Argument

To study the eqn. (3.8) just derived above, it is convenient at this time to discuss the evaluation of integrals like the one in (3.8) for asymptotically large times.

As we have already said in Chapter Two, for an integral of the type

$$f(t) = \int_{\alpha}^{\beta} g(x)\cos[h(x)t]dx \tag{3.9}$$

where g is continuous and h is twice continuously differentiable we can make approximations to it for small t and large t. For small times the argument of the cosine is approximately constant over the range of integration and consequently the maximum contribution to the integral comes from the region $x \sim \eta$ where g has its maximum value. Thus for $t \sim 0$

$$f(t) \sim \cos[h(\eta)t] \int_{\alpha}^{\beta} g(x)dx \quad . \tag{3.10}$$

For large times the major contribution to the integral comes from the neighborhood of the endpoints α and β, of the interval and any points at which $h'(x) = 0$; the latter being more important in the first approximation [3]. Contributions from other regions cancel by destructive interference.

Suppose first that λ is the only stationary point, i.e.: $h'(\lambda) = 0$, $\alpha < \lambda < \beta$ and assume that $h''(\lambda) > 0$. Expanding h about $x = \lambda$

$$h(x) - h(\lambda) = u^2 \ . \tag{3.11}$$

With this change of variable (3.9) becomes

$$f(t) \sim \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} g(x)\cos[h(x)t]dx$$

$$= \int_{-u_1}^{u_2} 2u \frac{g(x)}{h'(x)} \cos[t(h(\lambda) + u^2)]du \tag{3.12}$$

where

$$\begin{aligned} u_1 &= [h(\lambda-\varepsilon) - h(\lambda)]^{1/2} \\ u_2 &= [h(\lambda+\varepsilon) - h(\lambda)]^{1/2} \ . \end{aligned} \tag{3.13}$$

We have assumed that only the region where $u \sim 0$ contributes to the integral. For $u \sim 0$ we may let

$$\begin{aligned} g(x) &\rightarrow g(\lambda) \\ \frac{2u}{h'(x)} &\rightarrow \left[\frac{2}{h''(\lambda)}\right]^{1/2} \end{aligned} \tag{3.14}$$

so that

$$f(t) \sim \left[\frac{2}{h''(\lambda)}\right]^{1/2} g(\lambda) \int_{-u_1}^{u_2} \cos[t(u^2 + h(\lambda))]du \ . \tag{3.15}$$

By the same argument as above we may let u_1 and u_2 go to infinity and we get after integration

$$F(t) \sim \left[\frac{2\pi}{th''(\lambda)}\right]^{1/2} g(\lambda)\cos\left[th(\lambda) + \frac{\pi}{4}\right] . \tag{3.16}$$

This is the result of Kelvin's Stationary Phase Argument.

The contributions arising from the endpoints are much more difficult to obtain. We will merely state the result given in [3]. Assume that

(1) we have broken up the integration interval and made changes of variable so that $h(x)$ is monotonically increasing in each subinterval $\alpha \leq x \leq \beta$,

(2) $g(x)$ is continuously differentiable for $\alpha \leq x \leq \beta$,

(3) $h(x)$ is differentiable,

(4) $h'(x) = (x-\alpha)^{\rho-1}(\beta-x)^{\sigma-1}h_1(x)$

where $\rho \geq 1$, $\sigma \geq 1$ and $h_1(x)$ is positive and continuously differentiable for $\alpha \leq x \leq \beta$, and

(5) $0 < \nu \leq 1$, $0 < \mu \leq 1$ where $1-\nu$ and $1-\mu$ are the orders of possible singularities of the integrand at α and β respectively.

Then to first order

$$\int_\alpha^\beta g(x)(x-\alpha)^{\nu-1}(\beta-x)^{\mu-1}\cos[h(x)t]dx$$
$$\sim \left(\frac{k(0)}{\rho}\Gamma\left(\frac{\nu}{\rho}\right)\frac{\cos[h(\alpha)t + \frac{\pi}{2}\frac{\nu}{\rho}]}{t^{\nu/\rho}} - \frac{\ell(0)}{\sigma}\Gamma\left(\frac{\mu}{\sigma}\right)\frac{\cos[h(\beta)t - \frac{\pi}{2}\frac{\mu}{\sigma}]}{t^{\mu/\sigma}}\right) \tag{3.17}$$

where

$$k(u) = g_1(x(u))u^{1-\nu} \frac{dx}{du}$$
$$u^{\rho} = h(x) - h(\alpha) \tag{3.18}$$

in the first term and

$$\ell(v) = g_1(x(v))v^{1-\mu} \frac{dx}{dv}$$
$$v^{\sigma} = h(\beta) - h(x) \tag{3.19}$$

in the second term and

$$g_1(x) = g(x)(x-\alpha)^{\nu-1}(\beta-x)^{\mu-1} \quad . \tag{3.20}$$

Notice that if $h'(x) \neq 0$ anywhere in $\alpha \leq x \leq \beta$ then by assumption (4) $\rho = \sigma = 1$. Furthermore if $g_1(x)$ is bounded and continuously differentiable in $\alpha \leq x \leq \beta$ then $\nu = \mu = 1$ and the contribution of the endpoints to the integral is $O(t^{-1})$. On the other hand the contribution of any stationary points was found in eqn. (3.16) to be $\sim t^{-\frac{1}{2}}$.

§3.3 Example

With the results of Section 3.2 in hand we now look at some examples of eqn. (3.8). We set $r = 4$, so that (using the above notation)

$$h(x) = [1 + \tfrac{1}{4}(x^2 - 4)^2]^{1/2} \quad . \tag{3.21}$$

$h(x)$ has a stationary point at $\lambda = 2$ as shown in the right-hand diagrams of Fig. 3.1. To illustrate the results of Section 3.2 we take three cases of eqn.(3.8) with different integration intervals.

Case (a): $\lambda=2 < \alpha=2.5 < \beta=5$

Case (b): $\alpha=1 < \lambda=2 < \beta=3$

Case (c): $\alpha=\frac{1}{2} < \beta=1 < \lambda=2$

as shown in Fig. 3.1 (a), (b), and (c) respectively. Case (b) does and cases (a) and (c) do not include the stationary point in their intervals.

Note that δ must be large to have large oscillations. Therefore, in light of the fact that we want $\beta = 5$ in case (a), let us interchange the limits and change the sign of the integral in eqn.(3.8). Since case (a) is similar to case (c), we will discuss only cases (b) and (c) at length.

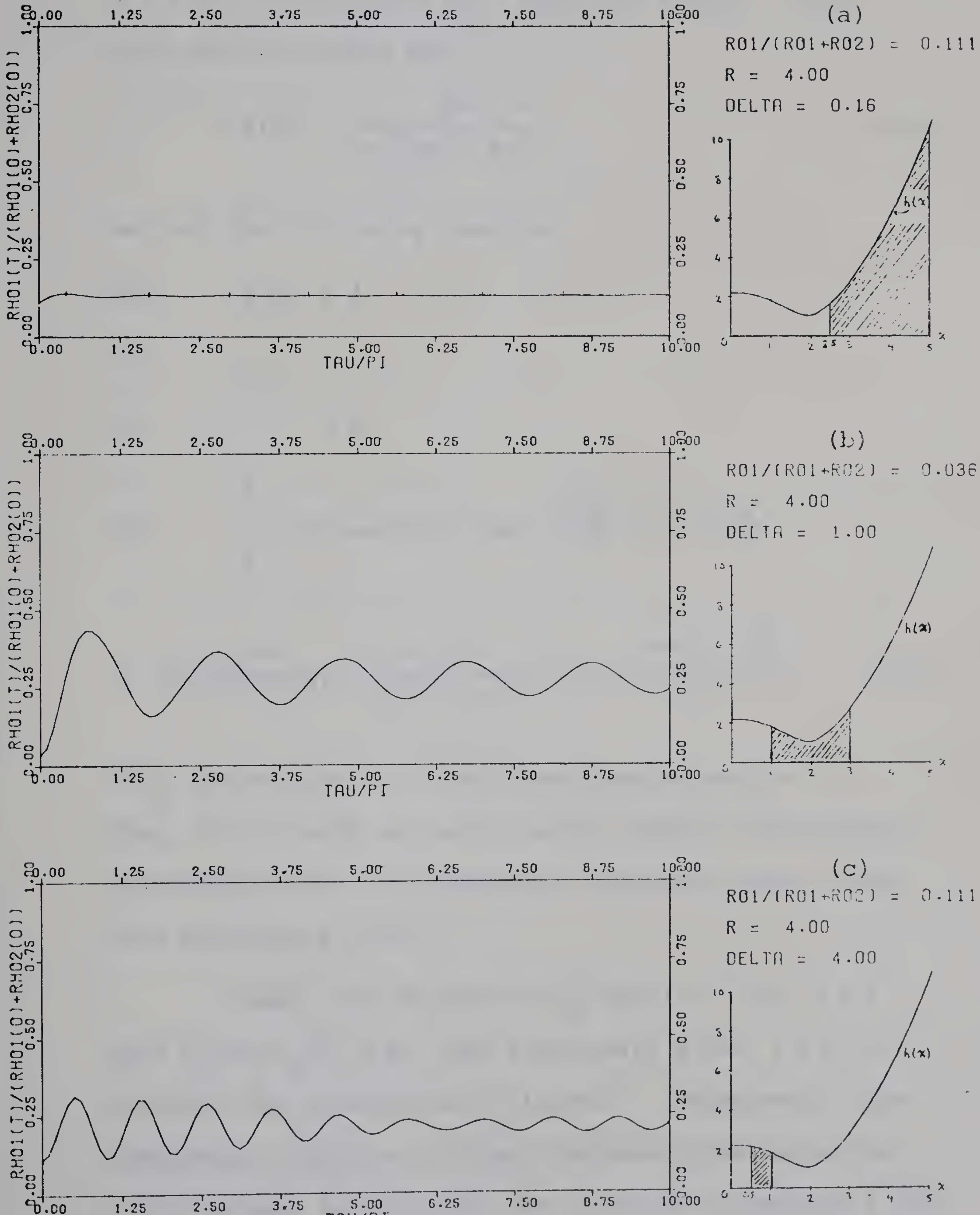


Fig. 3.1. Examples of zero temperature oscillations.

Case (b): To get $\alpha = 1$ we let $\delta = 1$ and thus $\beta = 3$ if $\rho_2(0)/\rho_1(0) = 27$. Applying result (3.16) with h(x) as above and

$$g(x) = \frac{x^2}{1 + \frac{1}{4}(x^2 - 4)^2} \tag{3.22}$$

we get the following results:

(1) $g(\lambda) = 4$

(2) $h(\lambda) = 1$

(3) $h''(\lambda) = 4$

(4) $\int_1^3 g(x)\cos[h(x)\tau]dx \sim \sqrt{\frac{8\pi}{\tau}} \cos(\tau + \frac{\pi}{4})$

$$\therefore \frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} \sim \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} - 0.27 \frac{\cos(\tau + \frac{\pi}{4})}{\sqrt{\tau}} . \tag{3.23}$$

This expression is a very good approximation to Fig. 3.1(b) even as early as the second oscillation, verifying that the asymptotic frequency comes from the stationary point.

Case (c): To have $\alpha = \frac{1}{2}$ and $\beta = 1$, set $\delta = 4$ and $\rho_2(0)/\rho_1(0) = 8$. The stationary point $\lambda = 2$ is outside the integration interval. Consequently the asymptotic behavior of the time evolution equation (3.8) arises mainly from the integral's endpoints and is found using result (3.17). Following the steps of

the argument:

(1) $g(x) = g_1(x) = \dfrac{x^2}{1 + \frac{1}{4}(x^2 - 4)^2}$ is continuously differentiable for $\alpha \leq x \leq \beta$

$$g(\alpha{=}\tfrac{1}{2}) = \frac{16}{289} \qquad g(\beta{=}1) = \frac{4}{13}$$

(2) $h(x) = -[1 + \frac{1}{4}(x^2 - 4)^2]^{1/2}$

$$h(\alpha) = -\frac{17}{8} \qquad h(\beta) = -\frac{\sqrt{13}}{2}$$

(3) $h'(x) = h_1(x) = -\dfrac{x(x^2 - 4)}{2\sqrt{1 + \frac{1}{4}(x^2 - 4)^2}}$ is positive and continuously differentiable for $\alpha \leq x \leq \beta$

$$h'(\alpha) = \frac{15}{34} \qquad h'(\beta) = \frac{3}{\sqrt{13}}$$

(4) $\rho = \sigma = \nu = \mu = 1$

(5) $\displaystyle\int_{\frac{1}{2}}^{1} g(x)\cos[h(x)\tau]dx$

$$\sim \{k(0)\cos[h(\alpha)\tau + \tfrac{\pi}{2}] - \ell(0)\cos[h(\beta)\tau - \tfrac{\pi}{2}]\}/\tau$$

(6) To determine $k(0)$ use eqn. (3.18):

$$k(0) = g_1(\alpha)/\frac{du}{dx}\Big|_{x=\alpha} = \frac{g(\alpha)}{h'(\alpha)} = \frac{32}{255} \sim .1255$$

(7) Determine $\ell(0)$ using eqn. (3.19):

$$\ell(0) = g_1(\beta)/\frac{dv}{dx}\Big|_{x=\beta} = -\frac{g(\beta)}{h'(\beta)} = -\frac{4}{3\sqrt{13}} \sim -.37$$

(8) Thus

$$\int_{\frac{1}{2}}^{1} = \{\frac{32}{255}\cos[\frac{17}{8} - \frac{\pi}{2}] + \frac{4}{3\sqrt{13}}\cos[\frac{\sqrt{13}}{2}\tau + \frac{\pi}{2}]\}/\tau$$

The time evolution equation becomes

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} \sim \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} - \{0.167\ \sin(2.125\tau) - 0.49\ \sin(1.8\tau)\}/\tau. \quad (3.24)$$

The frequencies arising from the two endpoints cause beats of frequency

$$\omega = \frac{\omega_1 - \omega_2}{2} \sim 0.1625 \quad (3.25)$$

or period $T = \frac{2\pi}{\omega} \sim 12.3\,\pi$. The phase or average frequency is $\omega = \omega_1+\omega_2/2 \sim 1.96$ with corresponding period $T \sim 1.02\,\pi$. This value for the phase period is only approximate since the amplitudes in (3.24) are not equal as was assumed to derive (3.25). In fact the period should be greater since the longer period sinusoidal has the greater amplitude.

All these conclusions are borne out in Fig. 3.1(c) and Fig. 3.2. Fig. 3.2 best shows the beat phenomenon and also the fact that (3.24) is a good approximation almost from time $\tau = 0$.

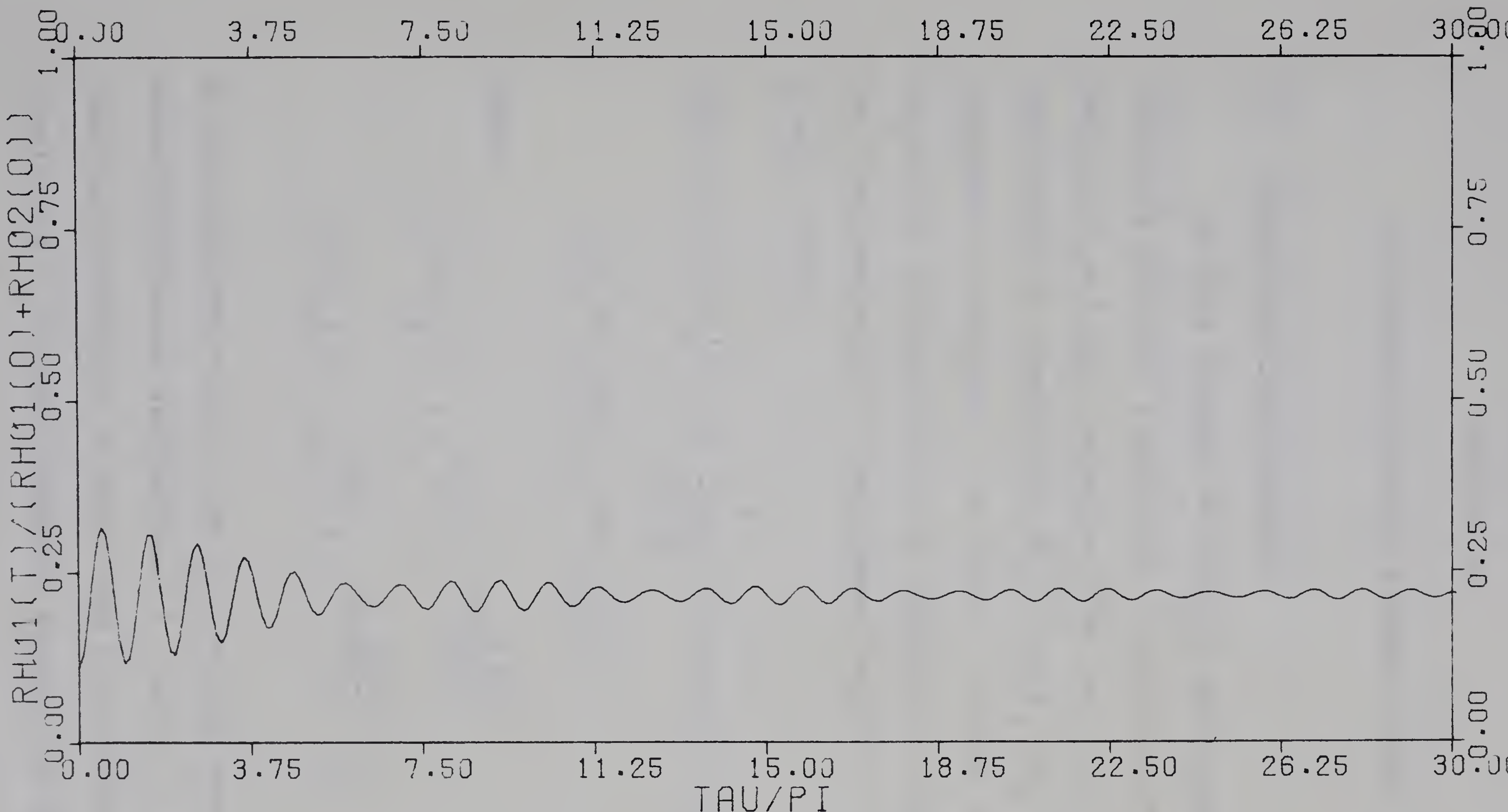


Fig. 3.2. Plot of Fig. 3.1(c) for $0 \leq \tau \leq 30\pi$ ($\rho_2(0)/\rho_1(0) = 8.$; $r = 4.$; $\delta = 4.$) This shows the beats of period $T = 12.3\pi$ arising from interference of the endpoint contribution to the integral in eqn. (3.8).

CHAPTER IV

TWO TEMPERATURE-TWO CHANNEL SYSTEM

§4.1 Time Evolution Equation

In this chapter we study a two-channel system initially with the a particle in equilibrium at temperature $T_a = 1/k_B\beta_1$ separate from the b particles in equilibrium at temperature $T_b = 1/k_B\beta_2$ (k_B is Boltzmann's constant). At time t = 0 the particles mix and an external field is switched on spontaneously. The time evolution of the number density is described by eqn. (1.21)

$$n_k^{(1)}(t) = n^{(o)}(\varepsilon_k^{(1)}) + \frac{1}{2}\,\frac{V_o^2}{V_o^2+\xi_k^2}\,[1 - \cos(\varepsilon_k^{(1)} - \varepsilon_k^{(2)})\,\frac{t}{\hbar}] \times$$

$$\times\ \{n^{(o)}(\varepsilon_k^{(2)}) - n^{(o)}(\varepsilon_k^{(1)})\} \qquad (4.1)$$

where

$$\begin{aligned} n^{(o)}(\varepsilon_k^{(1)}) &= e^{\beta_1\varepsilon_{c_1}}\, e^{-\beta_1 \frac{\hbar^2 k^2}{2m_1}} \\ n^{(o)}(\varepsilon_k^{(2)}) &= e^{\beta_2\varepsilon_{c_2}}\, e^{-\beta_2\left(\frac{\hbar^2 k^2}{2m_2} + \varepsilon_o\right)} \, . \end{aligned} \qquad (4.2)$$

We have assumed that T_a and T_b are high enough to use Maxwell-Boltzmann statistics. ε_{c_1} and ε_{c_2} are the chemical potentials of the a and b particles respectively and ε_o is the threshold energy for the formation

of particles b.

It is convenient to first calculate $\rho_1(0)$, $\rho_2(0)$, and $\rho_2(0)/\rho_1(0)$, where

$$\rho_i(0) = \frac{1}{(2\pi)^3}\int 4\pi\, k^2 dk\, n^{(o)}(\varepsilon_k^{(i)}) \qquad i = 1,2 \tag{4.3}$$

Using the fact that

$$\int_0^\infty k^2 e^{-\delta k^2} dk = \frac{\sqrt{\pi}}{4}\,\frac{1}{\delta^{3/2}} \tag{4.4}$$

we get

$$\begin{aligned}
\rho_1(0) &= e^{\beta_1 \varepsilon_{c_1}} \left(\frac{m_1}{2\pi\beta_1\hbar^2}\right)^{3/2} \\
\rho_2(0) &= e^{\beta_2(\varepsilon_{c_2}-\varepsilon_o)} \left(\frac{m_2}{2\pi\beta_2\hbar^2}\right)^{3/2} \\
\frac{\rho_2(0)}{\rho_1(0)} &= \frac{e^{\beta_2(\varepsilon_{c_2}-\varepsilon_o)}}{e^{\beta_1\varepsilon_{c_1}}} \left(\frac{m_2}{m_1}\,\frac{\beta_1}{\beta_2}\right)^{3/2} .
\end{aligned} \tag{4.5}$$

Using expressions (4.5) calculate

$$\begin{aligned}
n^{(o)}(\varepsilon_k^{(2)}) - n^{(o)}(\varepsilon_k^{(1)}) \\
&= e^{\beta_2(\varepsilon_{c_2}-\varepsilon_o)}\, e^{-\beta_2 \frac{\hbar^2k^2}{2m_2}} - e^{\beta_1\varepsilon_{c_1}}\, e^{-\beta_1\frac{\hbar^2k^2}{2m_1}} \\
&= \left(\frac{2\pi\beta_1\hbar^2}{m_1}\right)^{3/2} \rho_1(0) \left[\frac{\rho_2(0)}{\rho_1(0)}\,\gamma^{3/2} e^{-\gamma\beta_1\frac{\hbar^2k^2}{2m_1}} - e^{-\beta_1\frac{\hbar^2k^2}{2m_1}}\right]
\end{aligned} \tag{4.6}$$

where

$$\gamma = \frac{m_1}{m_2}\frac{\beta_2}{\beta_1} \quad . \tag{4.7}$$

Thus the time evolution of the a particle density is given by

$$\rho_1(t) = \frac{1}{(2\pi)^3}\int d^3k \; n_k^{(1)}(t)$$

$$= \rho_1(0) + \frac{1}{(2\pi)^2}\int k^2 dk \; \frac{V_o^2}{V_o^2+\xi_k^2}\left[1-\cos(\lambda_k^{(1)}-\lambda_k^{(2)})\frac{t}{\hbar}\right] \times$$

$$\times \quad \{n^{(o)}(\varepsilon_k^{(2)}) - n^{(o)}(\varepsilon_k^{(1)})\} \quad . \tag{4.8}$$

Define substitutions similar to (1.24) of Chapter One. That is

$$\delta = \frac{\mu}{m_1}\beta_1 V_o \qquad r = \frac{\varepsilon_o}{V_o} \qquad \tau = \frac{2V_o t}{\hbar} \qquad \mu = \frac{m_1 m_2}{m_2 - m_1}$$

$$x^2 = \frac{\hbar^2 k^2}{2\mu V_o} \tag{4.9}$$

Using the definitions (4.9) and dividing by $\rho_1(0)+\rho_2(0)$, eqn. (4.8) for the a particle density becomes finally:

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} = \frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)}\Bigg(1 + \frac{2}{\sqrt{\pi}}\,\delta^{3/2}\int_0^\infty \frac{x^2 dx}{1+\frac{1}{4}(x^2-r)^2} \times$$

$$\times \left[1-\cos\tau\sqrt{1+\tfrac{1}{4}(x^2-r)^2}\right] \times$$

$$\times \left\{\gamma^{3/2}\frac{\rho_2(0)}{\rho_1(0)}e^{-\gamma\delta x^2} - e^{-\delta x^2}\right\}\Bigg) \tag{4.10}$$

§4.2 Examples

Before discussing any specific examples let us obtain the general behavior of the time evolution equation (4.10) for large and small times.

For large times we can apply the arguments of Section 3.2. The endpoint contributions to the integral of eqn. (4.10) are $O(\tau^{-3/2})$ since the integrand vanishes at $x = 0$ and $x = \infty$. Thus to lowest order we need only apply the Stationary Phase Argument to the stationary point at $\lambda = r^{1/2}$. Thus

$$g(x) = \frac{x^2}{1+\frac{1}{4}(x^2-r)^2}\left(\gamma^{3/2}\,\frac{\rho_2(0)}{\rho_1(0)}\,e^{-\gamma\delta x^2} - e^{-\delta x^2}\right) \tag{4.11}$$

$$h(x) = \sqrt{1+\frac{1}{4}(x^2-r)^2} \tag{4.12}$$

$$h''(\lambda = r^{1/2}) = r \tag{4.13}$$

Therefore

$$\int_0^\infty g(x)\,\mathrm{cox}[h(x)\tau]\,dx = \sqrt{\frac{2\pi r}{\tau}}\left(\gamma^{3/2}\,\frac{\rho_2(0)}{\rho_1(0)}\,e^{-\gamma\delta r} - e^{-\delta r}\right) \times$$

$$\times \cos(\tau + \tfrac{\pi}{4}) \tag{4.14}$$

Thus for large τ eqn. (4.10) becomes

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} = \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} - C\,\frac{\cos(\tau+\frac{\pi}{4})}{\sqrt{\tau}} \tag{4.15}$$

where

$$C = \frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} (8r\delta^3)^{1/2} \{\gamma^{3/2} \frac{\rho_2(0)}{\rho_1(0)} e^{-\gamma\delta r} - e^{-\delta r}\} \quad . \tag{4.16}$$

So as time progresses, oscillations of all other frequencies will die out faster than the oscillation with frequency $\omega=1$. In other words asymptotically the period of oscillation for any two channel system with any initial conditions is always $T = 2\pi$.

For small τ the behavior of eqn. (4.10) can be quite different. As was said in Section 3.2, the largest contribution to the time evolution equation then comes from the integration region where the non-oscillating factor of the integrand (i.e.: $g(x)$) has its maximum. Thus let us study $g(x)$ and obtain approximations to eqn. (4.10) for various limiting cases. Rewrite $g(x)$ as

$$g(x) = \tilde{g}_1(x) - \tilde{g}_2(x) \tag{4.17}$$

where

$$\tilde{g}_1(x) = \gamma^{3/2} \frac{\rho_2(0)}{\rho_1(0)} \frac{x^2}{1+\frac{1}{4}(x^2-r)^2} e^{-\gamma\delta x^2}$$

$$\tilde{g}_2(x) = \frac{x^2}{1+\frac{1}{4}(x^2-r)^2} e^{-\delta x^2} \quad . \tag{4.18}$$

Case I $\rho_2(0) << \rho_1(0)$

If $\rho_2(0) = 0$ then $\tilde{g}_1(x) = 0$ and eqn. (4.10) simplifies to eqn. (1.25). That is, we are back to the one temperature system of Chapter One. Consequently if $\rho_2(0) \ll \rho_1(0)$ we expect similar behavior, and so we now discuss the one temperature system and the $\rho_2(0) \ll \rho_1(0)$ system together.

There are two limiting cases. If δ is large then the main contribution to the integral comes from the region around $x = \delta^{-1/2}$ where the factor

$$x^2 e^{-\delta x^2} \tag{4.19}$$

attains its maximum value. Consequently the oscillations of eqn. (4.10) have frequency

$$\omega = [1 + \tfrac{1}{4}(\tfrac{1}{\delta} - r)^2]^{1/2} \tag{4.20}$$

for small times.

On the other hand if δ is small then $e^{-\delta x^2} \sim 1$ for not too large x and the main contribution to the integral is the region around $x = r^{1/2}$ where the factor

$$[1 + \tfrac{1}{4}(x^2 - r)^2]^{-1} \tag{4.21}$$

attains its maximum value. Then the oscillations of eqn. (4.10) have frequency $\omega = 1$ for all times.

Looking at Fig. 1.1, if we consider $\delta = 1$ large and $\delta = 0.1$ small then the above arguments plus Kelvin's stationary phase argument accurately give the frequencies for the four examples. For instance, the dashed and solid curves of the lower graph both have initial frequencies $\omega = 1$ equal to the asymptotic frequency. Thus these oscillations decay slowly. The dotted curve and the upper graph curve have initial frequencies $\omega \sim \sqrt{21}$ and $\omega = \sqrt{5}$ respectively which are far from the asymptotic frequency. Thus these oscillations decay quickly. In the upper graph we can see the asymptotic frequency $\omega = 1$ emerging as the frequency $\omega = \sqrt{5}$ decays away. Examples similar to these with $\rho_2(0)/\rho_1(0) \neq 0$ but small are given in Fig. 4.1.

Of course if $\rho_2(0) >> \rho_1(0)$ then $\tilde{g}_2(x)$ is negligible compared to $\tilde{g}_1(x)$ and we expect oscillations similar to the above but "inverted". The above arguments for the frequencies are unchanged except that δ is replaced by $\gamma\delta$. Examples of this are given in Fig. 4.2.

<u>Case II γ large with $r = \delta = 1$</u>

To study this case break up eqn. (4.10) into the parts

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} = \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} - I_1(t) + I_2(t) \qquad (4.22)$$

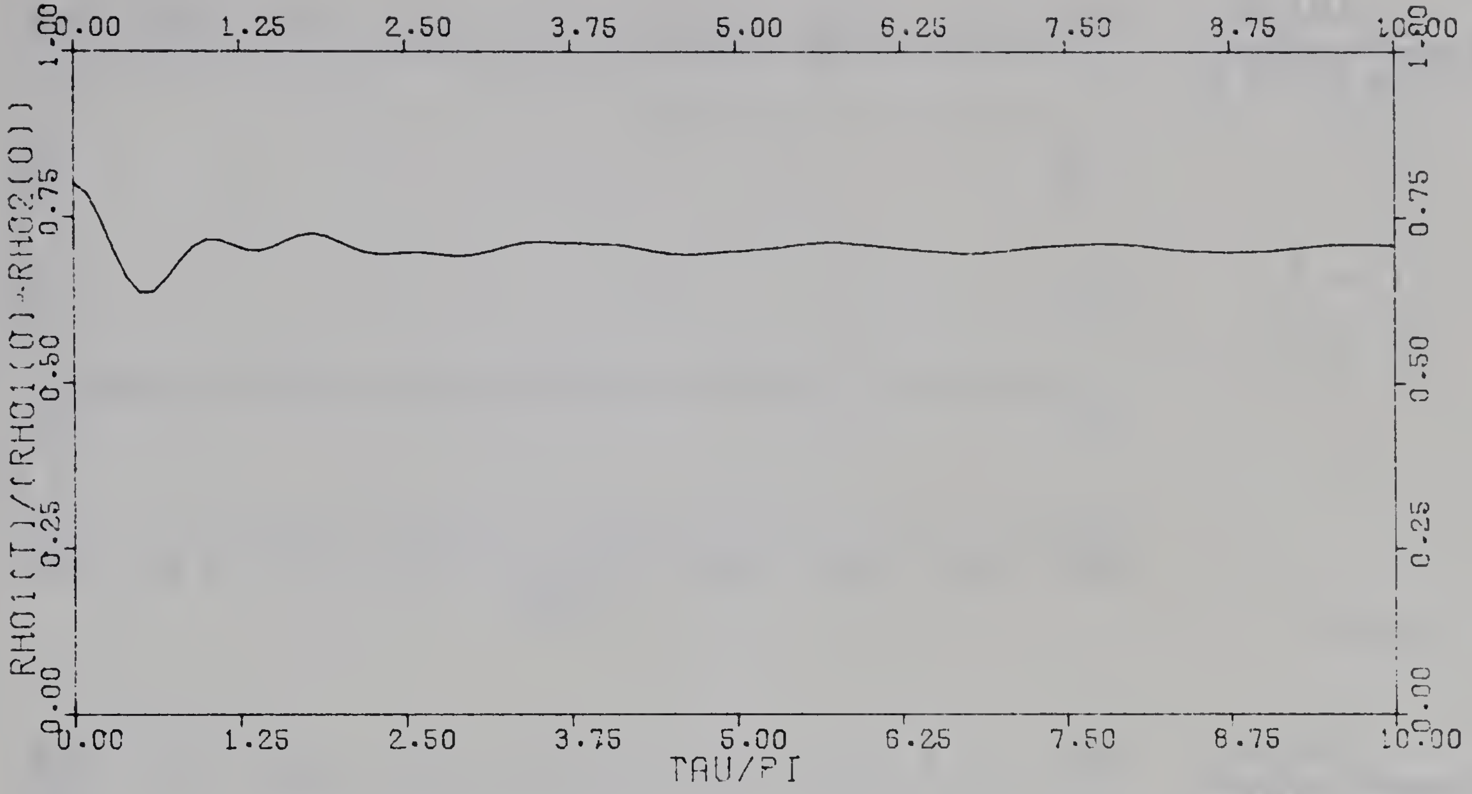


Fig. 4.1(a) $\dfrac{\rho_2(0)}{\rho_1(0)} = 0.25;\ r = 5.;\ \delta = 1.;\ \gamma = 3.$

(Compare this diagram to the upper graph of Fig. 1.1.)

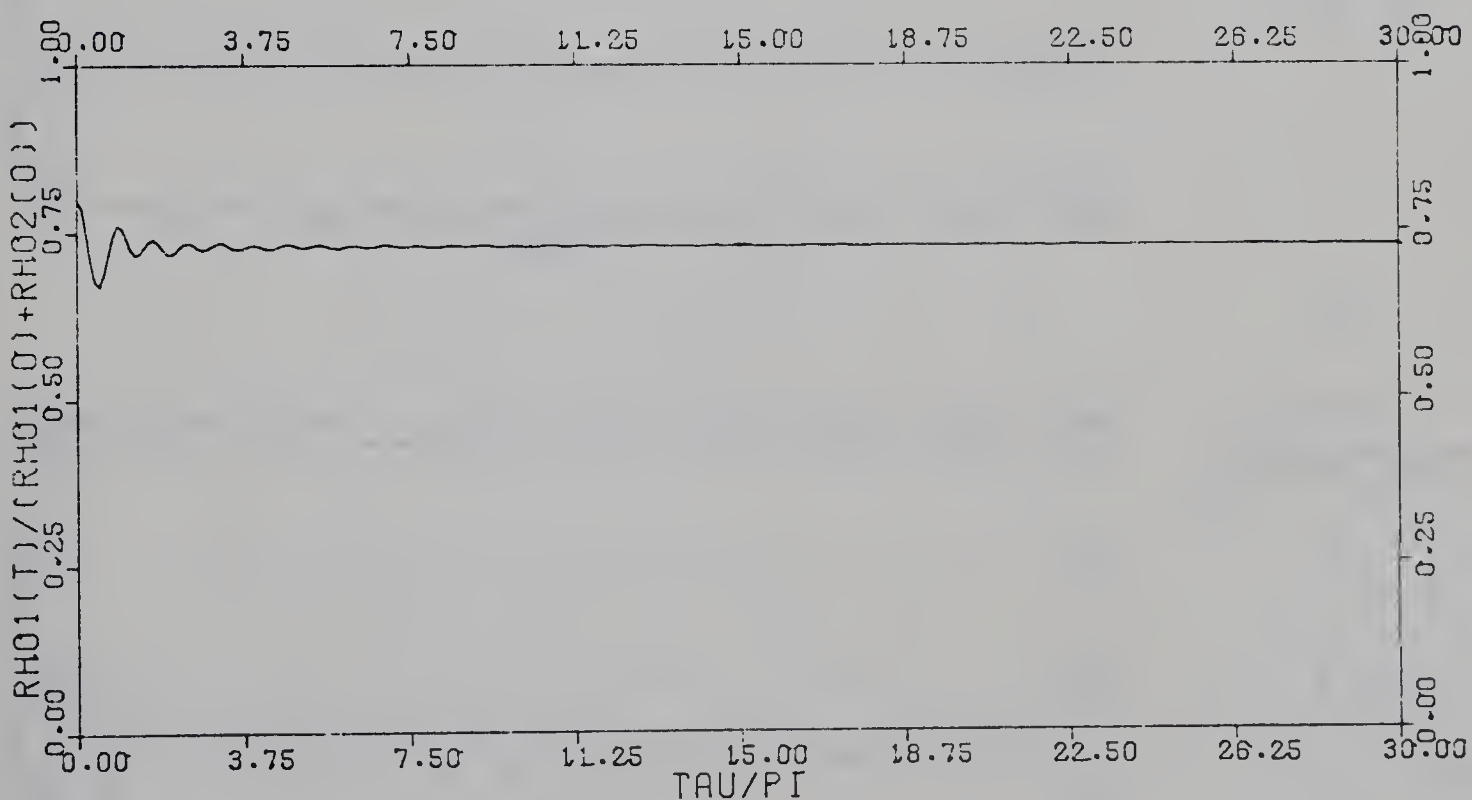
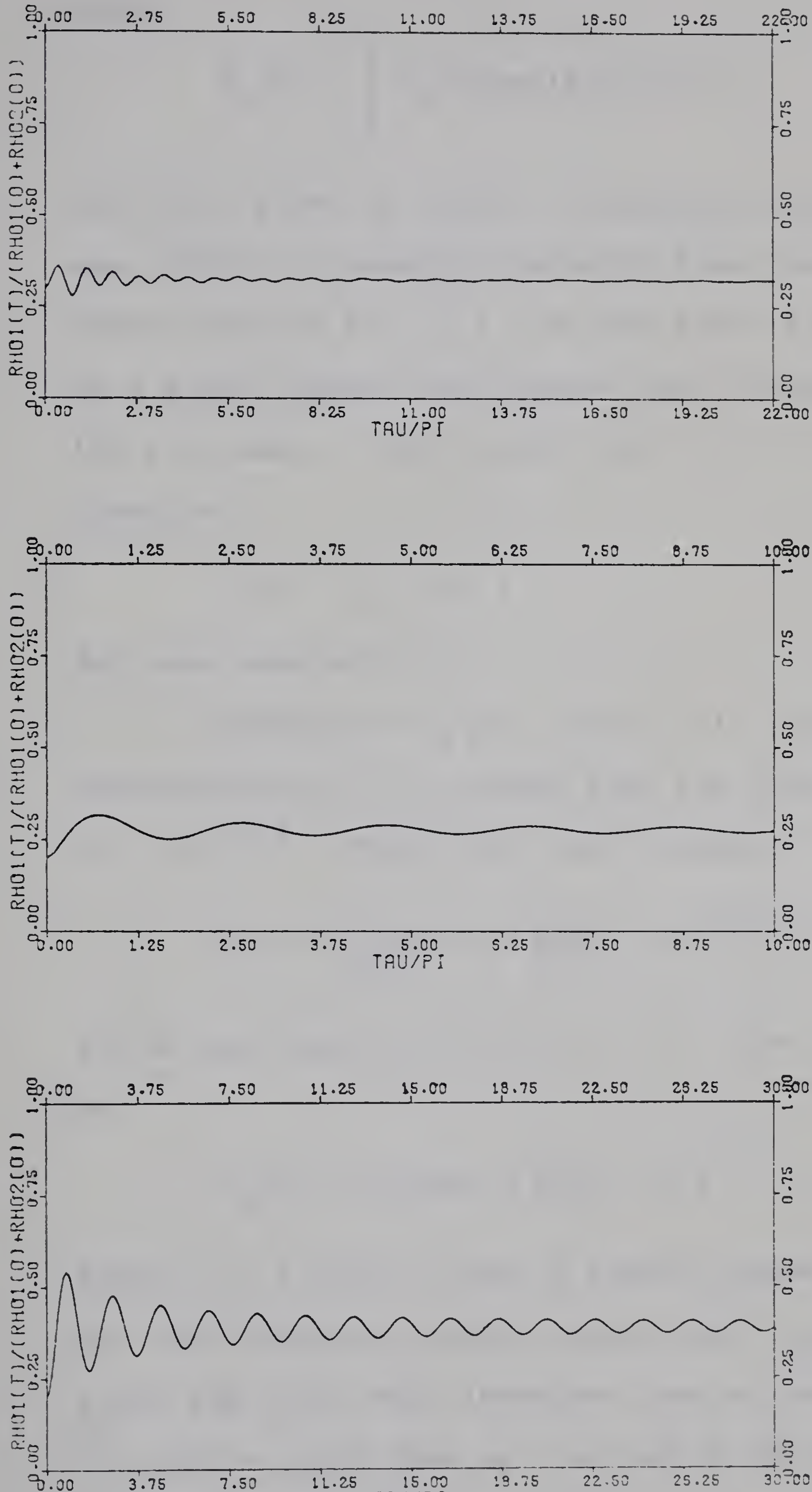


Fig. 4.1(b) $\dfrac{\rho_2(0)}{\rho_1(0)} = 0.25;\ r = 5.;\ \delta = 1.;\ \gamma = 0.5.$

Examples of the two temperature model with $\rho_2(0) << \rho_1(0)$.



(a)

$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} = 0.3$$

$r = 5.$

$\delta = 1.$

$\gamma = 2.$

(b)

$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} = 0.2.$$

$r = 0.$

$\delta = 0.1.$

$\gamma = 2.$

(c)

$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} = 0.2.$$

$r = 0.$

$\delta = 2.$

$\gamma = 0.5.$

Fig. 4.2. Examples of the two temperature model with $\rho_1(0) \ll \rho_2(0)$.

where

$$I_i(t) = \int_0^\infty \tilde{g}_i(x)\cos[h(x)\tau]dx \qquad (4.23)$$

with h(x) given in (4.12). Excluding $I_1(t)$, for $r = \delta = 1$ eqn. (4.22) is exactly the solid line curve of the lower graph of Fig. 1.1. By the above arguments it is a slowly damped oscillation with frequency $\omega_2 = 1$ for all times. Thus we can write $I_2(t)$ for limited times as

$$I_2(t) \sim A_2 \cos \tau \qquad (4.24)$$

for some constant A_2.

Looking at $I_1(t)$; since $\gamma\delta$ is large, the major contribution to $I_1(t)$ comes from the region around $x = (\gamma\delta)^{-1/2}$. Thus $I_1(t)$ has frequency

$$\omega_1 = h\left(\frac{1}{\sqrt{\gamma\delta}}\right) = \left[1 + \frac{1}{4}\left(\frac{1}{\gamma\delta} - r\right)^2\right]^{1/2} \qquad (4.25)$$

and we can write $I_1(t)$ for $\delta = r = 1$ for limited times as

$$I_1(t) \sim A_1 \cos\left[1 + \frac{1}{4}\left(\frac{1}{\gamma\delta} - 1\right)^2\right]^{1/2} \tau \quad . \qquad (4.26)$$

Since $\omega_1 \sim 1$ this is also a slowly damped oscillation but with slightly shorter period than $I_2(t)$. Thus $I_2(t)$ and $I_1(t)$ will interfere causing beats. In fact if $\rho_2(0) = \rho_1(0)$ then $A_1 \sim A_2$ and we can use the trigonometric identity

$$\cos\alpha - \cos\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\beta-\alpha) \tag{4.27}$$

to write (4.22) as

$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} \sim \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} + A'\sin\left(\frac{\omega_1+\omega_2}{2}\right)\tau\ \sin\left(\frac{\omega_1-\omega_2}{2}\right)\tau\ . \tag{4.28}$$

For example if $\gamma = 10$ then

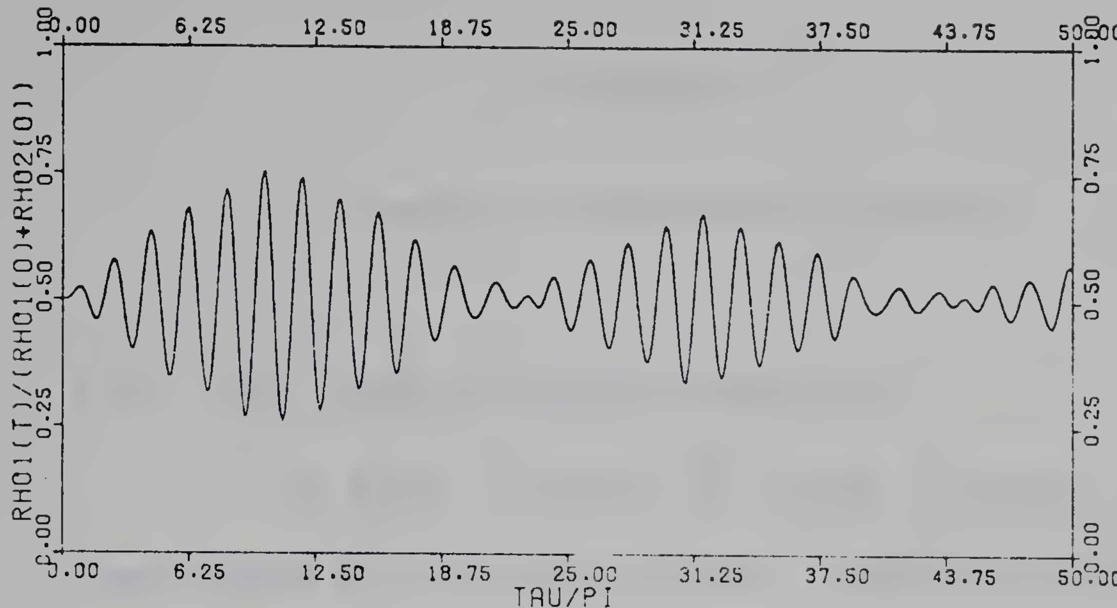
$$\frac{\rho_1(t)}{\rho_1(0)+\rho_2(0)} \quad \frac{\rho_1(\infty)}{\rho_1(0)+\rho_2(0)} + A'\ \sin(1.047\tau)\sin(0.047\tau). \tag{4.29}$$

This function describes Fig. 4.3(a) very well. The envelope period is

$$T_e = \frac{2\pi}{\omega_e} = \frac{2\pi}{0.047} \sim 42.5\ \pi \tag{4.30}$$

as the figure verifies. Other examples of this phenomenon are shown in Figs. 4.3(b) and (c).

(a)



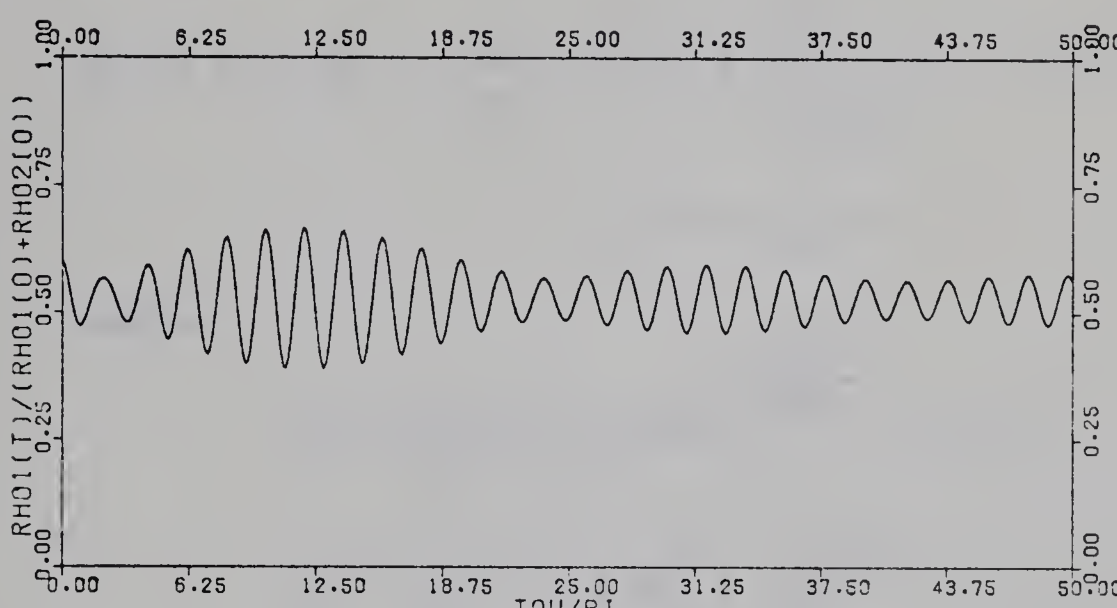
$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} = 0.5.$$

$r = 1.$

$\delta = 1.$

$\gamma = 10.$

(b)



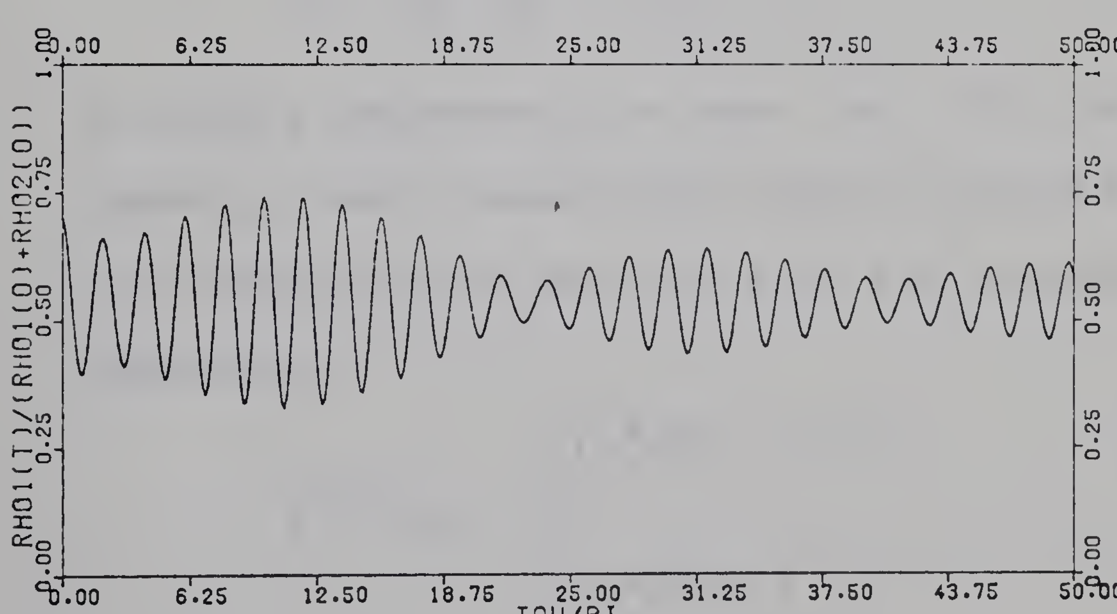
$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} = 0.6$$

$r = 1.$

$\delta = 1.$

$\gamma = 5.$

(c)



$$\frac{\rho_1(0)}{\rho_1(0)+\rho_2(0)} =0.7.$$

$r = 1.$

$\delta = 1.$

$\gamma = 10.$

Fig. 4.3. Examples of the two-temperature model with γ large.

CHAPTER V

WEAKLY DEGENERATE SYSTEM

§5.1 The Time Evolution Equation

In this chapter we study a weakly degenerate two-channel system of either bosons or fermions. We start with eqn. (1.23):

$$\rho_1(t) = \rho_1(0) - \frac{1}{(2\pi)^3}\int d^3k\ \frac{1}{2}\,\frac{V_o^2}{V_o^2+\xi_k^2}\,[1-\cos(\lambda_k^{(1)}-\lambda_k^{(2)})\tfrac{t}{\hbar}]\ \times$$

$$\times\ n^{(o)}(\varepsilon_k^{(1)}) \qquad (5.1)$$

where

$$n^{(o)}(\varepsilon_k^{(1)}) = [e^{-\beta\mu}e^{\beta\frac{\hbar^2k^2}{2m_1}} \pm 1]^{-1} \qquad \frac{\text{F-D}}{\text{B-E}}$$

$$\lambda_k^{(1)}-\lambda_k^{(2)} = 2(V_o^2+\xi_k^2)^{1/2}$$

$$\xi_k = \frac{1}{2}(\varepsilon_k^{(1)}-\varepsilon_k^{(2)})\ .$$

By weakly degenerate we mean that the system approximately obeys Maxwell-Boltzmann statistics. In the diagram below are plots of the distribution functions

$$n^{(o)}(\varepsilon) = \begin{cases} [e^{\alpha}e^{\beta\varepsilon} \pm 1]^{-1} & \frac{\text{F-D}}{\text{B-E}} \\ [e^{\alpha}e^{\beta\varepsilon}]^{-1} & \text{M-B} \end{cases} \qquad (5.2)$$

(where $\alpha = -\beta\mu$) as functions of α (for fixed $\beta\varepsilon$).

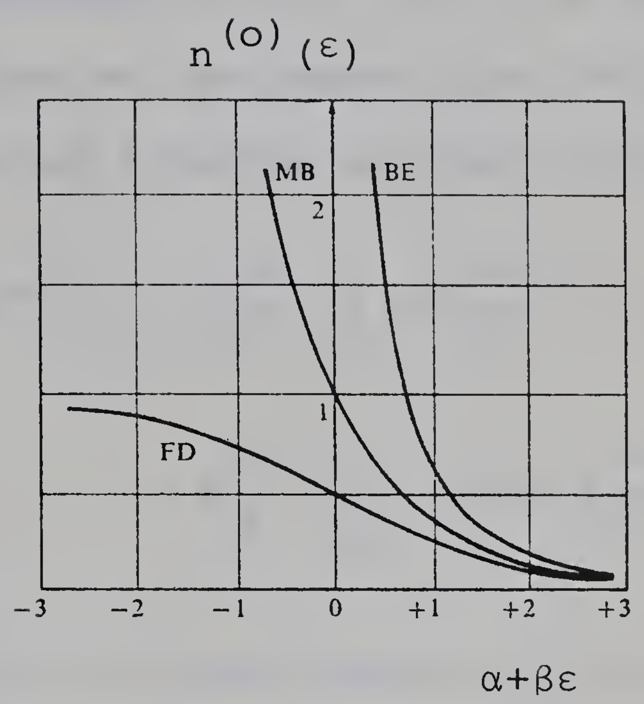


Fig. 5.1. Fermi-Dirac, Bose-Einstein, and Maxwell-Boltzmann distribution functions as functions of $\alpha+\beta\varepsilon$ ($\beta\varepsilon$ fixed arbitrarily at $\beta\varepsilon = 0$). Note that for α large and positive the three distributions are approximately equal. Physically this means that $n^{(o)}(\varepsilon) << 1$ and the effects of indistinguishability vanish. (Taken from [4]).

From Fig. 5.1 we conclude that α is large and positive for weakly degenerate systems. Using the abbreviation $Z = e^{\alpha} = e^{-\beta\mu}$ we have that Z is large. Thus we can expand the distribution functions for bosons or fermions as power series in 1/Z:

$$n^{(o)}(\varepsilon) = \frac{e^{-\beta\varepsilon}}{Z} \left(1 \pm \frac{e^{-\beta\varepsilon}}{Z}\right)^{-1} \qquad \frac{\text{F-D}}{\text{B-E}}$$

$$= \frac{e^{-\beta\varepsilon}}{Z} \sum_{n=0}^{\infty} (\mp 1)^n \frac{e^{-n\beta\varepsilon}}{Z^n} \qquad \frac{\text{F-D}}{\text{B-E}} \tag{5.3}$$

Using (5.3) we first express $\rho_1(0)$ as a power series in 1/Z.

$$\rho_1(0) = \int d^3k \; n^{(o)}(\varepsilon_k^{(1)})$$

$$= \frac{2}{(2\pi)^2} \int_0^{\infty} k^2 dk \, \frac{e^{-\beta\frac{\hbar^2k^2}{2m_1}}}{Z} \left[\sum_{n=0}^{\infty} (\mp 1)^n \frac{e^{-n\beta\frac{\hbar^2k^2}{2m_1}}}{Z^n}\right] \qquad \frac{\text{F-D}}{\text{B-E}}$$

$$= \left(\frac{m_1}{2\pi\beta\hbar^2}\right)^{3/2} \frac{1}{Z} \sum_{n=0}^{\infty} \frac{(\mp 1)^n}{Z^n} \frac{1}{(n+1)^{3/2}} \qquad \frac{\text{F-D}}{\text{B-E}} \tag{5.4}$$

Here we have used the fact that

$$\int_0^{\infty} dx \; x^2 \, e^{-\delta x^2} = \frac{\sqrt{\pi}}{4} \frac{1}{\delta^{3/2}} \; . \tag{5.5}$$

Define the (dimensionless) degeneracy parameter y:

$$y = \rho_1(0)\ \left(\frac{2\pi\beta\hbar^2}{m_1}\right)^{3/2} \quad . \tag{5.6}$$

Then, rewriting (5.4) in terms of (5.6) we get

$$y = \frac{1}{Z} \sum_{n=0}^{\infty} (\mp 1)^n \frac{1}{(n+1)^{3/2}} \left(\frac{1}{Z}\right)^n \qquad \frac{\text{F-D}}{\text{B-E}} \tag{5.7}$$

A simple reversion of series [5] gives a power series for 1/Z in terms of y. We get

$$\frac{1}{Z} = y\left[1 \pm \frac{y}{2^{3/2}} + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)y^2 + \ldots\right] \qquad \frac{\text{F-D}}{\text{B-E}} \tag{5.8}$$

Note that the greater the degeneracy, the larger the value of y; and also that y increases with increasing density and decreasing temperature.

Before we can proceed we must express the distribution functions (5.3) as power series in y. Using (5.8) and simply Cauchy multiplying the infinite series we get

$$\begin{aligned} n^{(o)}(\varepsilon) &= \frac{1}{Z} \sum_{n=0}^{\infty} \frac{(\mp 1)^n}{Z^n} \quad e^{-(n+1)\beta\varepsilon} \\ &= y\ e^{-\beta\varepsilon} \pm y^2 \left(\frac{e^{-\beta\varepsilon}}{2^{3/2}} - e^{-2\beta\varepsilon}\right) \\ &\quad + y^3 \left(e^{-3\beta\varepsilon} - \frac{1}{\sqrt{2}}\, e^{-2\beta\varepsilon} + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right) e^{-\beta\varepsilon}\right) + \ldots \end{aligned} \tag{5.9}$$

Using expression (5.6) for $1/\rho_1(0)$ and the dimensionless parameter substitutions (1.24) from the two-

channel system of Chapter One we get a time evolution equation similar to (1.25) but with one more parameter - the degeneracy, y. That is

$$\frac{\rho_1(t)}{\rho_1(o)} = 1 - \frac{2}{\sqrt{\pi}}\,\delta^{3/2}\int_0^\infty \frac{x^2dx}{1+\frac{1}{4}(x^2-r)^2}\,[1-\cos\tau\sqrt{1+\tfrac{1}{4}(x^2-r)^2}]$$

$$\times\ \{e^{-\delta x^2} \pm y(\frac{1}{2^{3/2}}\,e^{-\delta x^2} - e^{-2\delta x^2}) + \ldots \qquad (5.10)$$

where

$$y = \rho_1(0)\left(\frac{2\pi\beta\hbar^2}{m_1}\right)^{3/2}$$

$$\tau = \frac{2V_o t}{\hbar}$$

$$\delta = \frac{\mu}{m_1}\,\beta V_o \qquad (5.11)$$

$$r = \frac{\varepsilon_o}{V_o}$$

$$x^2 = \frac{\hbar^2k^2}{2\mu V_o}$$

and $\mu = \dfrac{m_1m_2}{m_2-m_1}$.

§5.2 Example

An example of a weakly degenerate two-channel system is shown in Fig. 5.2. It has the same parameters as the solid line curve of the lower graph of Fig. 1.1; that is, $r = \delta = 1$. The degeneracy parameter is $y = 0.2$.

The diagram shows that the oscillations for the Fermi-Dirac particles are smaller and the oscillations for the Bose-Einstein particles are larger in amplitude than those for the Maxwell-Boltzmann particles. The physical explanation for this is as follows.

Assume that the system is at a point in time where the density of the a particles is decreasing (e.g. at $\tau = 0.5\,\pi$) and the occupation of the states of the b particles is increasing. Referring to Fig. 5.1 we see that for fermions the occupation number, say of the b particles, is smaller for any given energy level due to the Pauli exclusion principle. Consequently the number of b particles that can be created must be less, and so the a to b transformation is inhibited. For bosons, a larger number of b particles can be present in states of any energy. Consequently the transformation of particles a into b is enhanced.

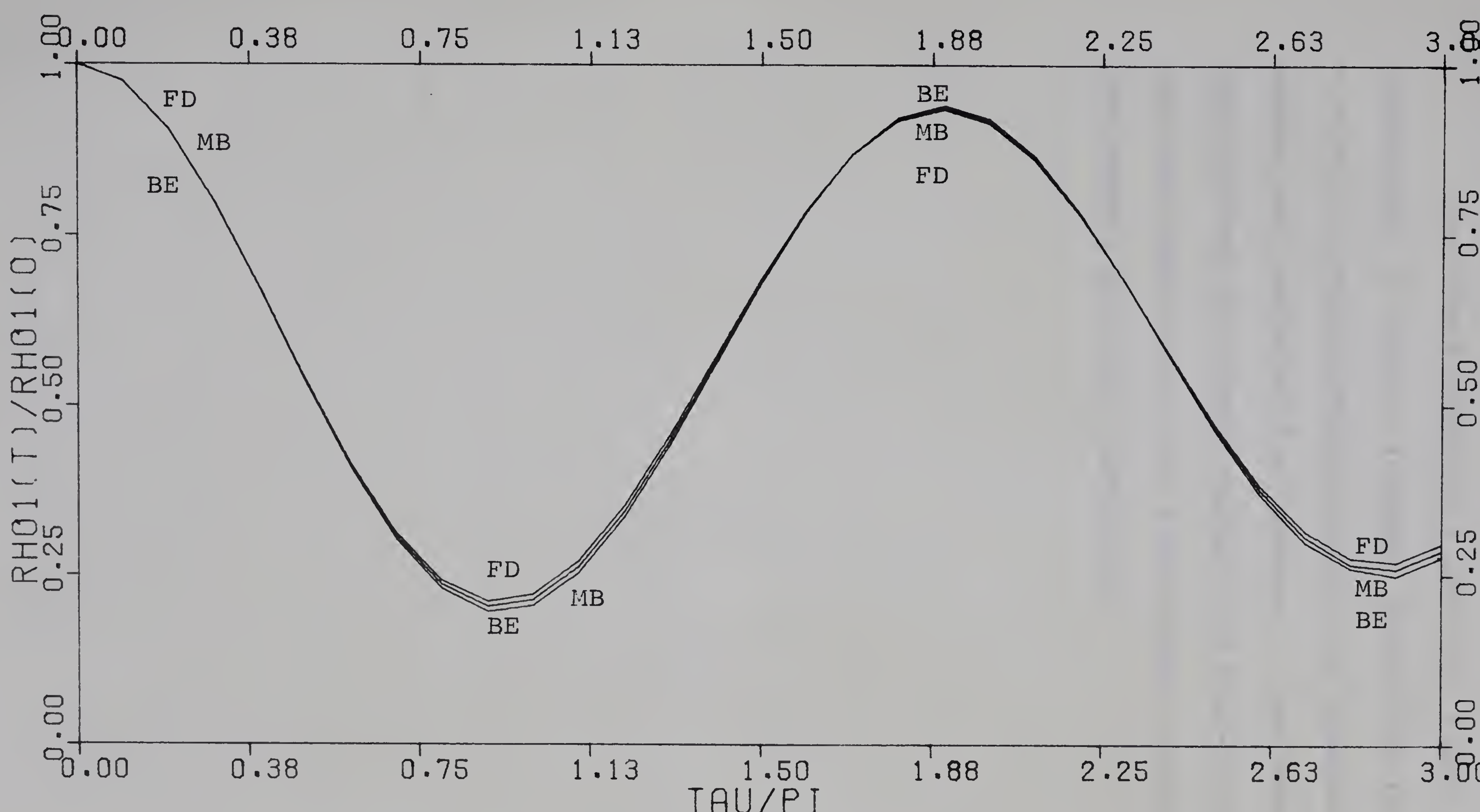
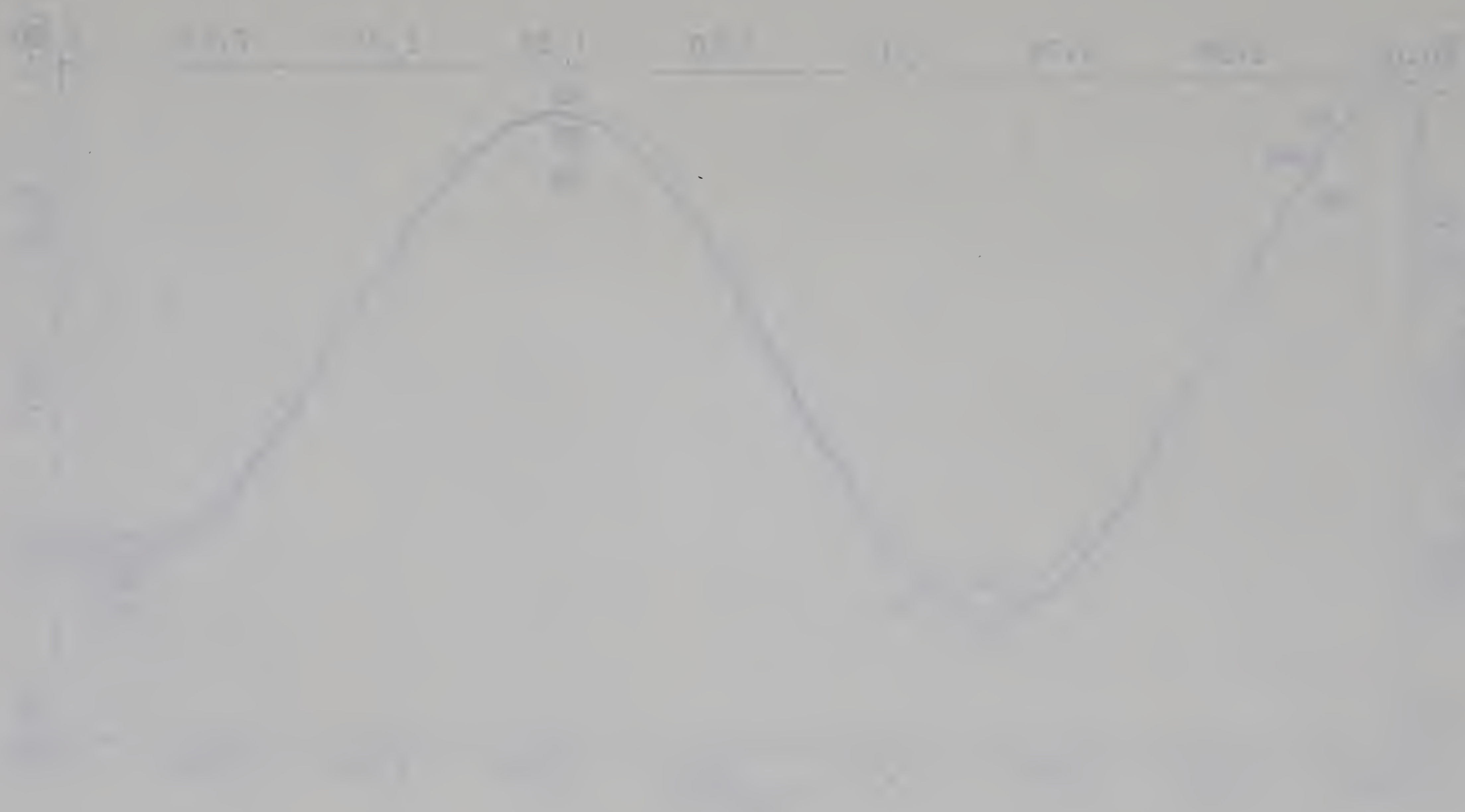


Fig. 5.2. r = 1.; δ = 1.; y = 0.2.; Density oscillations in a weakly degenerate system.



For times when the a particle density is increasing, a similar argument shows that the b to a particle transformations again are enhanced for bosons and inhibited for fermions. Thus the density oscillations are always larger for Bose-Einstein particles and smaller for Fermi-Dirac particles than for Maxwell-Boltzmann particles.

CONCLUSION

We have discussed several aspects of two and three channel interacting systems. Because in our model there are no collisions, there is no "Stosszahlansatz" to destroy the memory effect of the system. Thus we get damped oscillatory rather than exponential approach to equilibrium. We showed that in any n-channel system ($n > 1$) we get damped oscillations and that the period of these oscillations can increase greatly as the number of channels increases. Thus this model may explain some chemical oscillation reactions that have been studied experimentally.

We also discussed systems with interesting initial conditions, such as the zero temperature limit for fermions (Chapter III), two initial temperatures (Chapter IV) and moderately high densities or low temperatures (Chapter V). In each case we discussed the resulting time evolution equation from a mathematical standpoint, and wherever possible, gave physical explanations for the behavior of the system.

There are many aspects of this model that should still be studied. Some of these are: Bose-Einstein condensation, the highly-degenerate expansion, and the equilibrium properties of the model.

APPENDIX

For reference purposes, most of the graphs of the various time evolution equations done to this date (except some of those included in the body of the thesis) are assembled here. The integrals were solved numerically with an iterative Simpson rule. The n^{th} iteration partitioned the interval $0 \leq x \leq 4\delta^{-\frac{1}{2}}$ into 2^n subintervals. If the difference in the values of the integral computed in the n^{th} and $(n-1)^{th}$ iteration was less than 10^{-4} then the n^{th} iteration value was assumed to be correct.

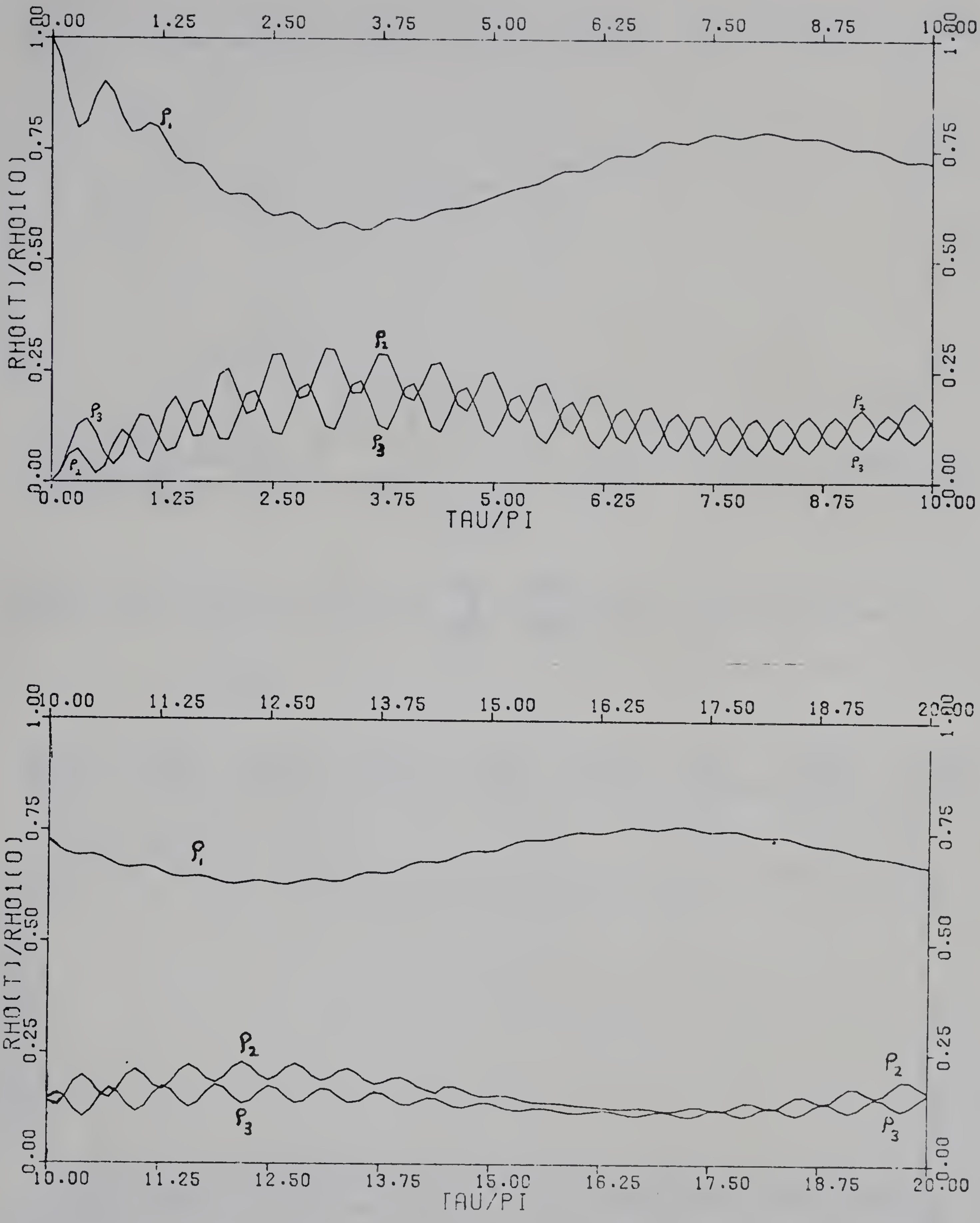


Fig. A.1. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 3.$; $\delta = 1.$

Upper graph: $0 \leq \tau \leq 10\,\pi$
Lower graph: $10\,\pi \leq \tau \leq 20\,\pi$.

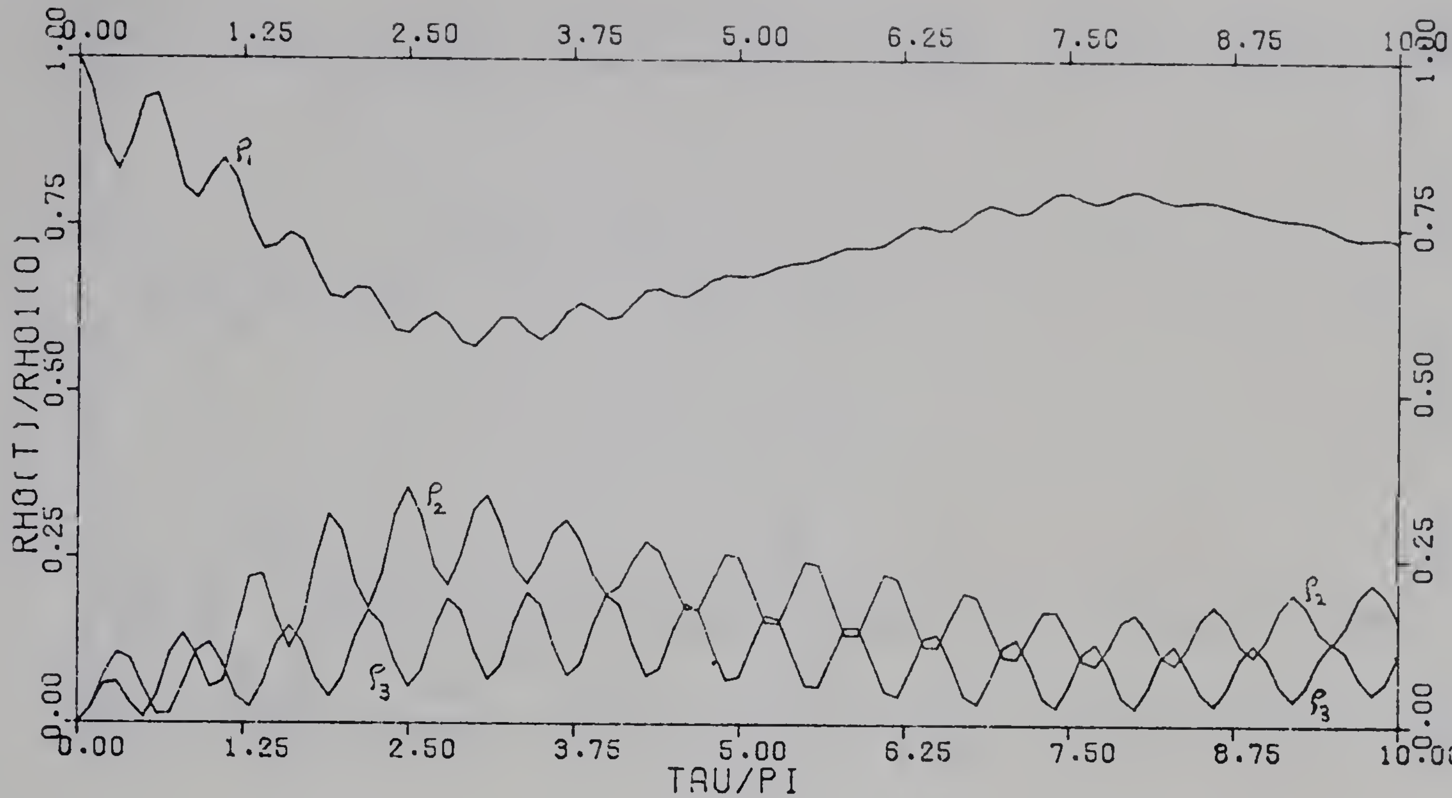


Fig. A.2. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = .5.$; $U_1 = 1.$; $U_2 = 3.$; $\delta = 1.$

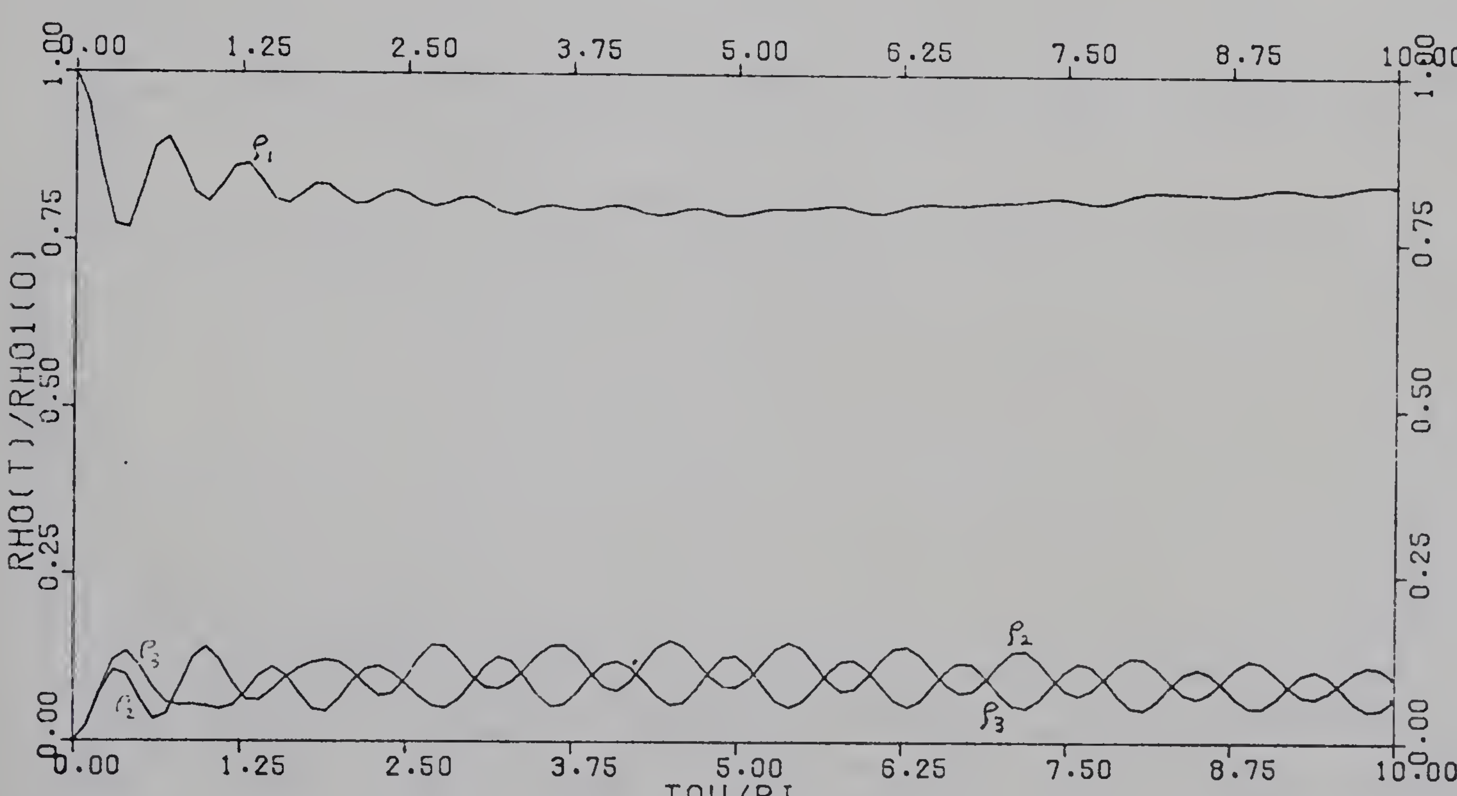


Fig. A.3. $r_2 = 4.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 2.$; $\delta = 1.$

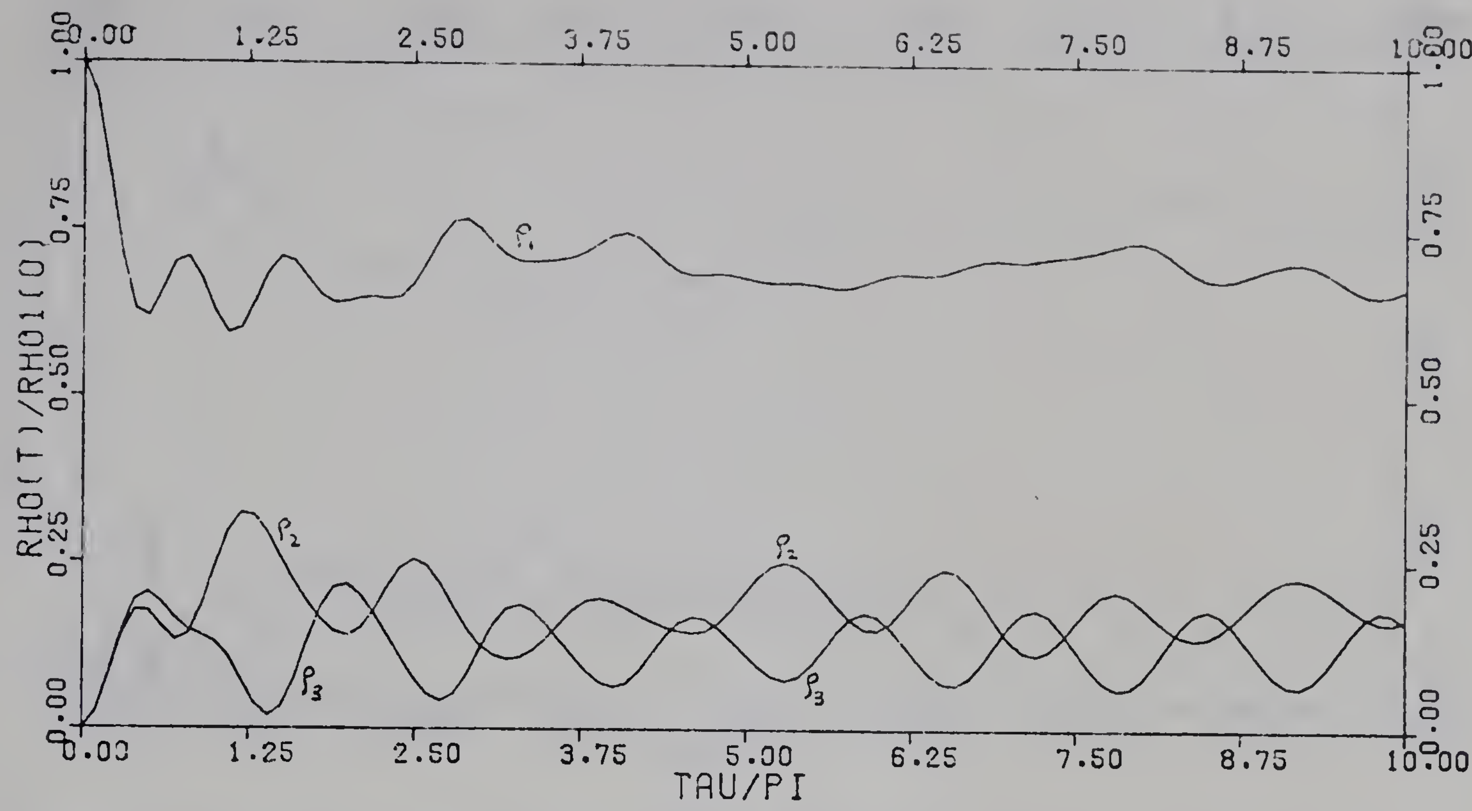


Fig. A.4. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = U_2 = 1.$;

$\delta = 1.$

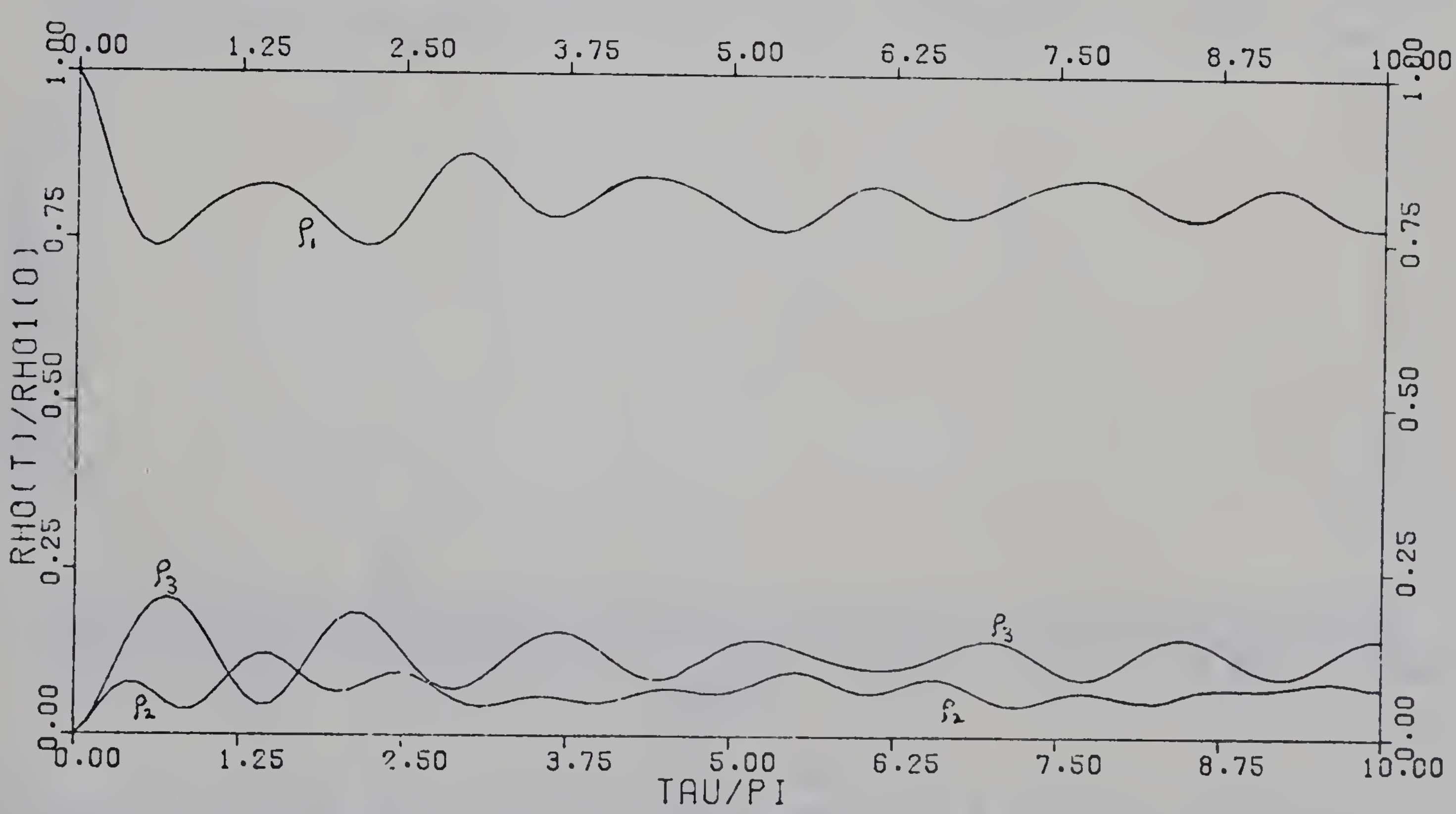


Fig. A.5. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = U_2 = 1.$;

$\delta = 0.1.$

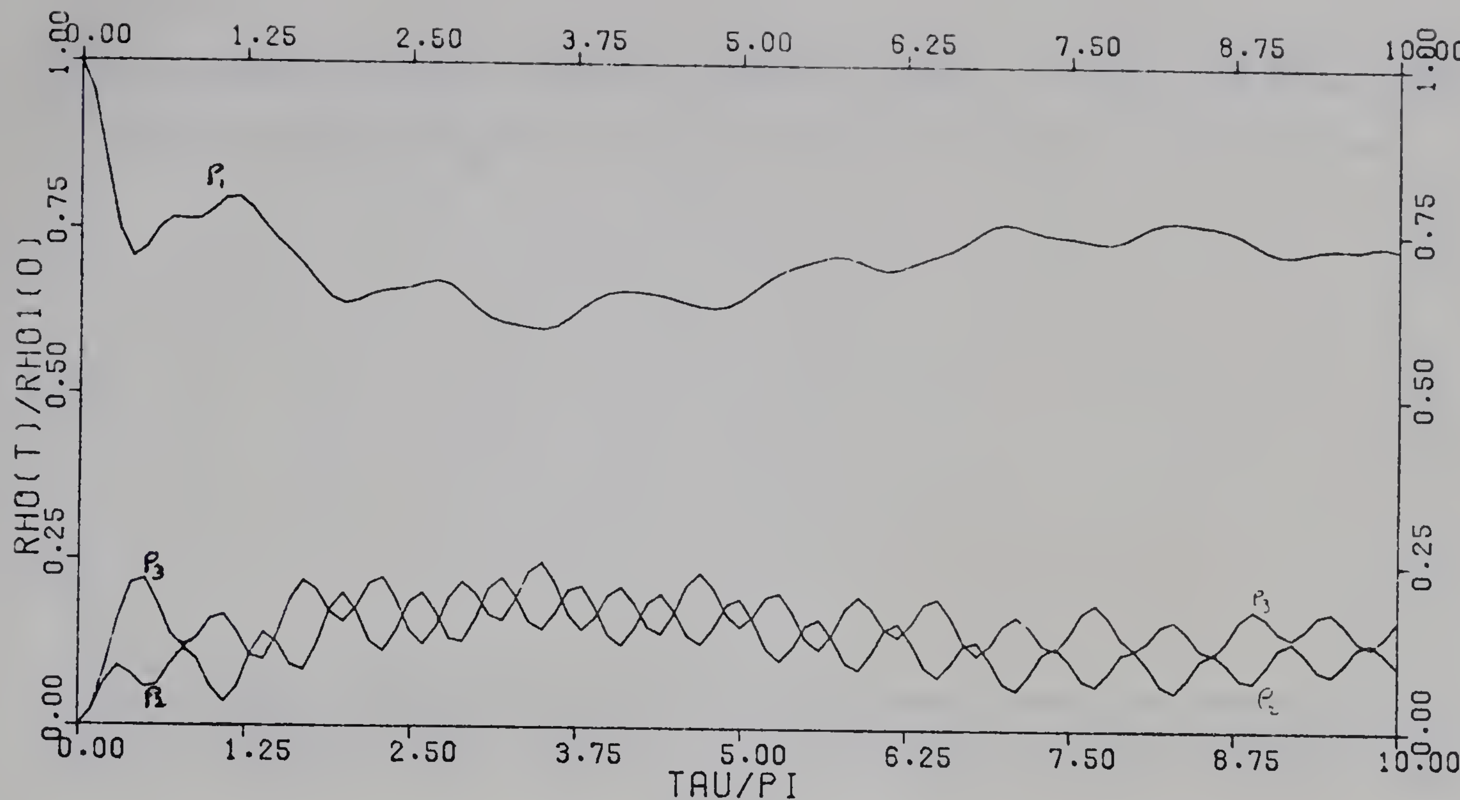


Fig. A.6. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 5.$;
$\delta = 0.5$.

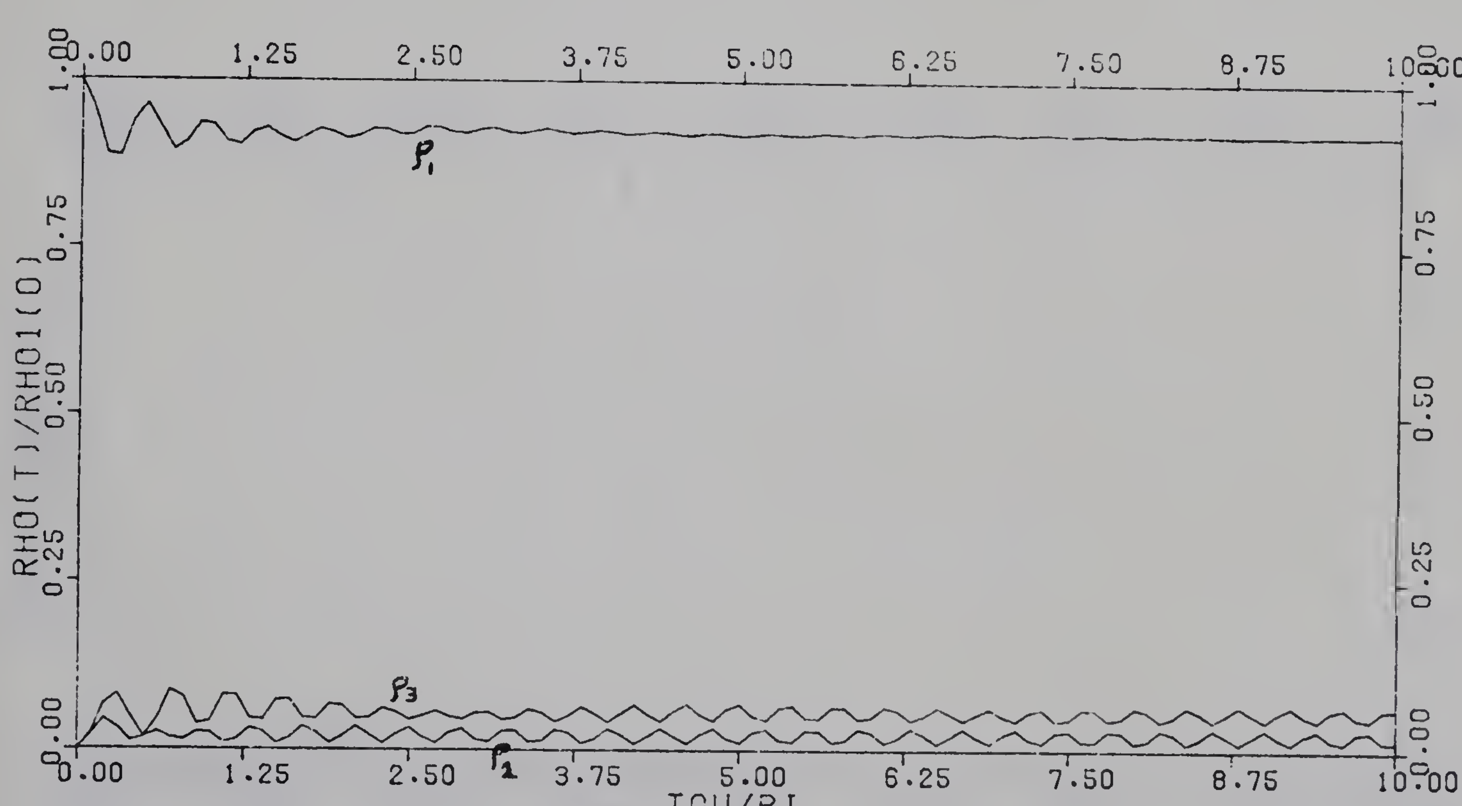


Fig. A.7. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 5.$;
$\delta = 1$.

Note that because U_2 is large, the ρ_1 oscillations damp out quickly.

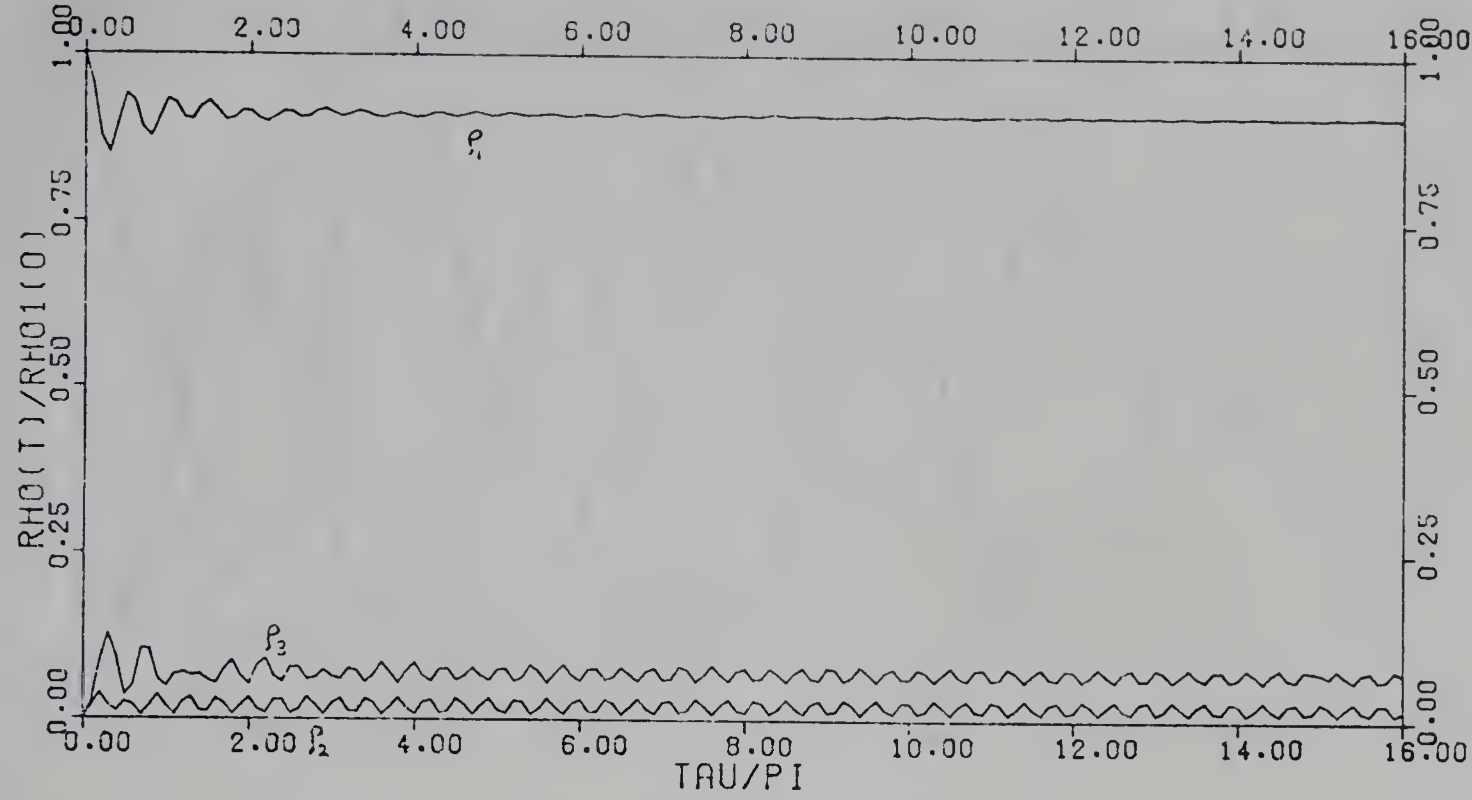


Fig. A.8. $r_2 = 1.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 5.$; $\delta = 1.$

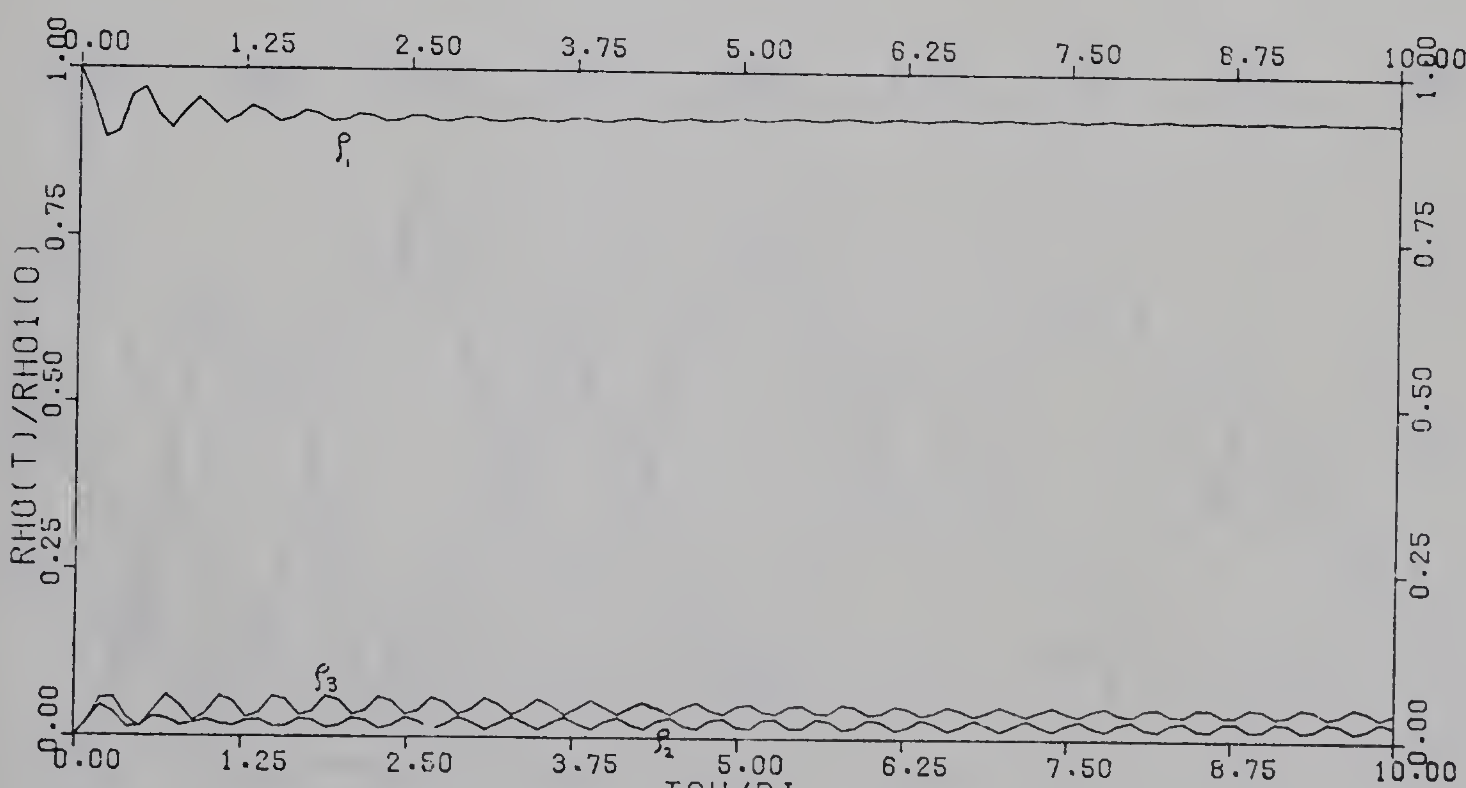


Fig. A.9. $r_2 = 4.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 1.$; $U_2 = 5.$; $\delta = 1.$

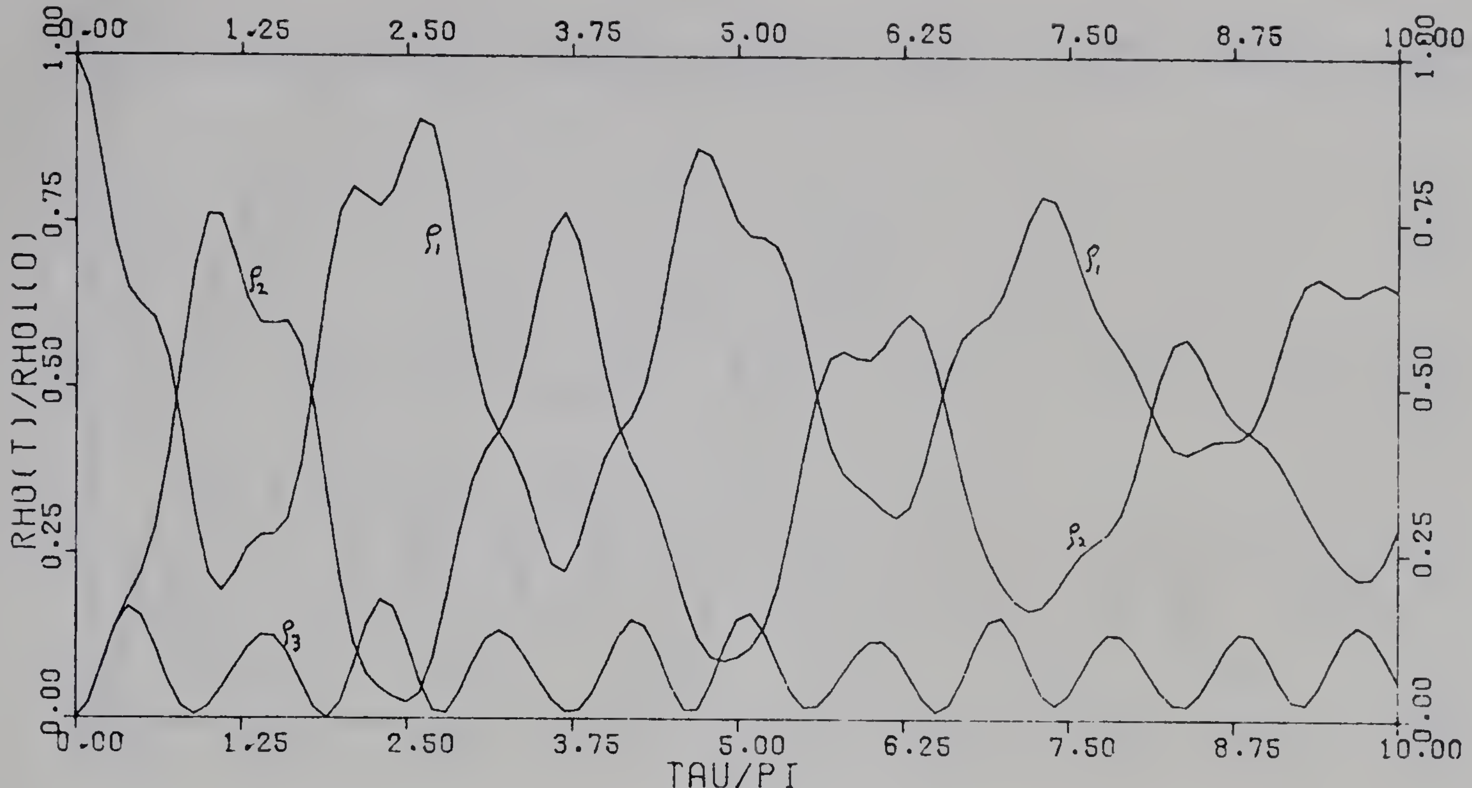


Fig. A.10. $r_2 = 1.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = U_2 = 1.$; $\delta = 5.$

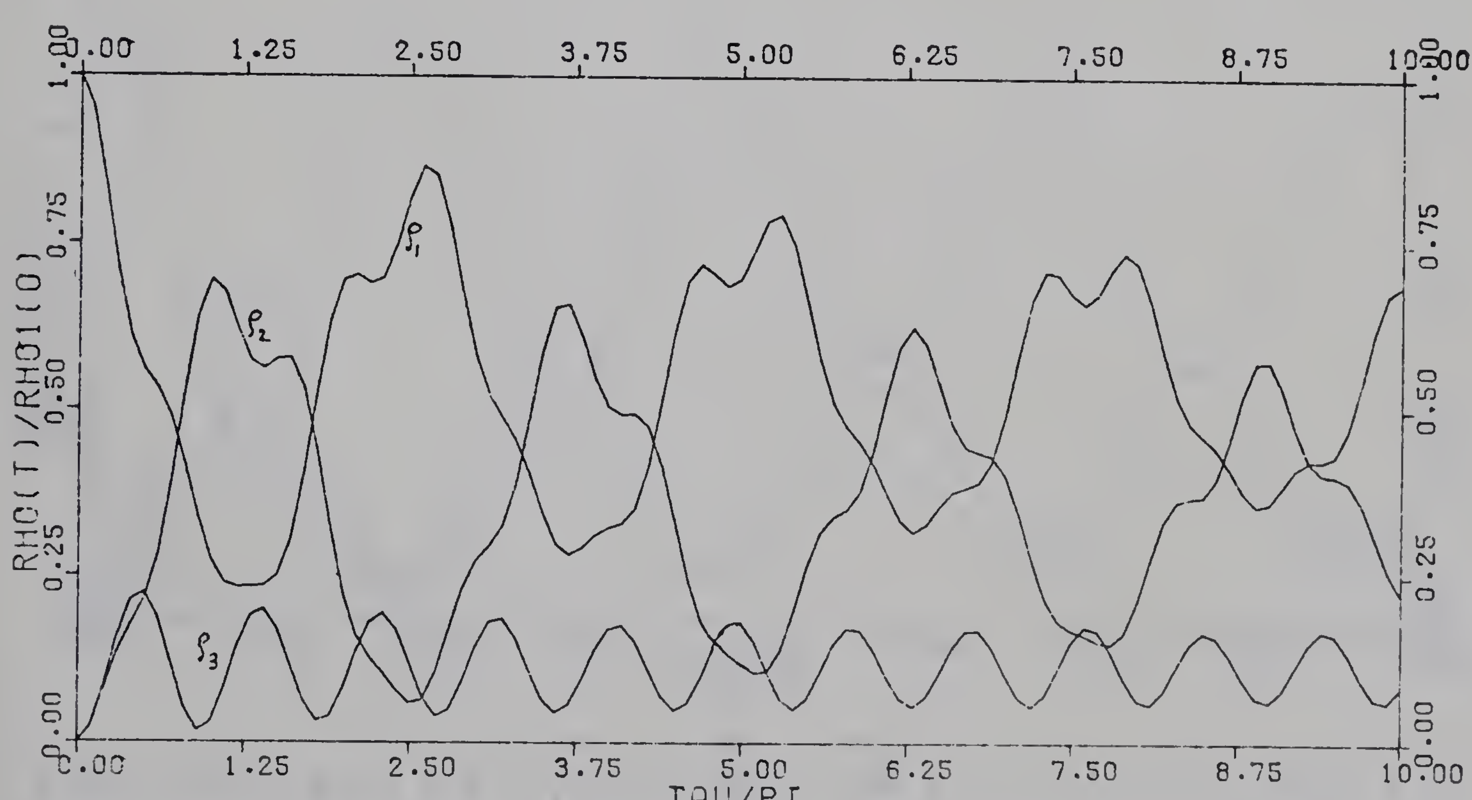


Fig. A.11. $r_2 = 1.$; $r_3 = 5.$; $\frac{m_1}{m_2} = 0.$; $\frac{m_1}{m_3} = 0.5.$; $U_1 = U_2 = \delta = 1.$
Note the similarity to Fig. 2.2(b).

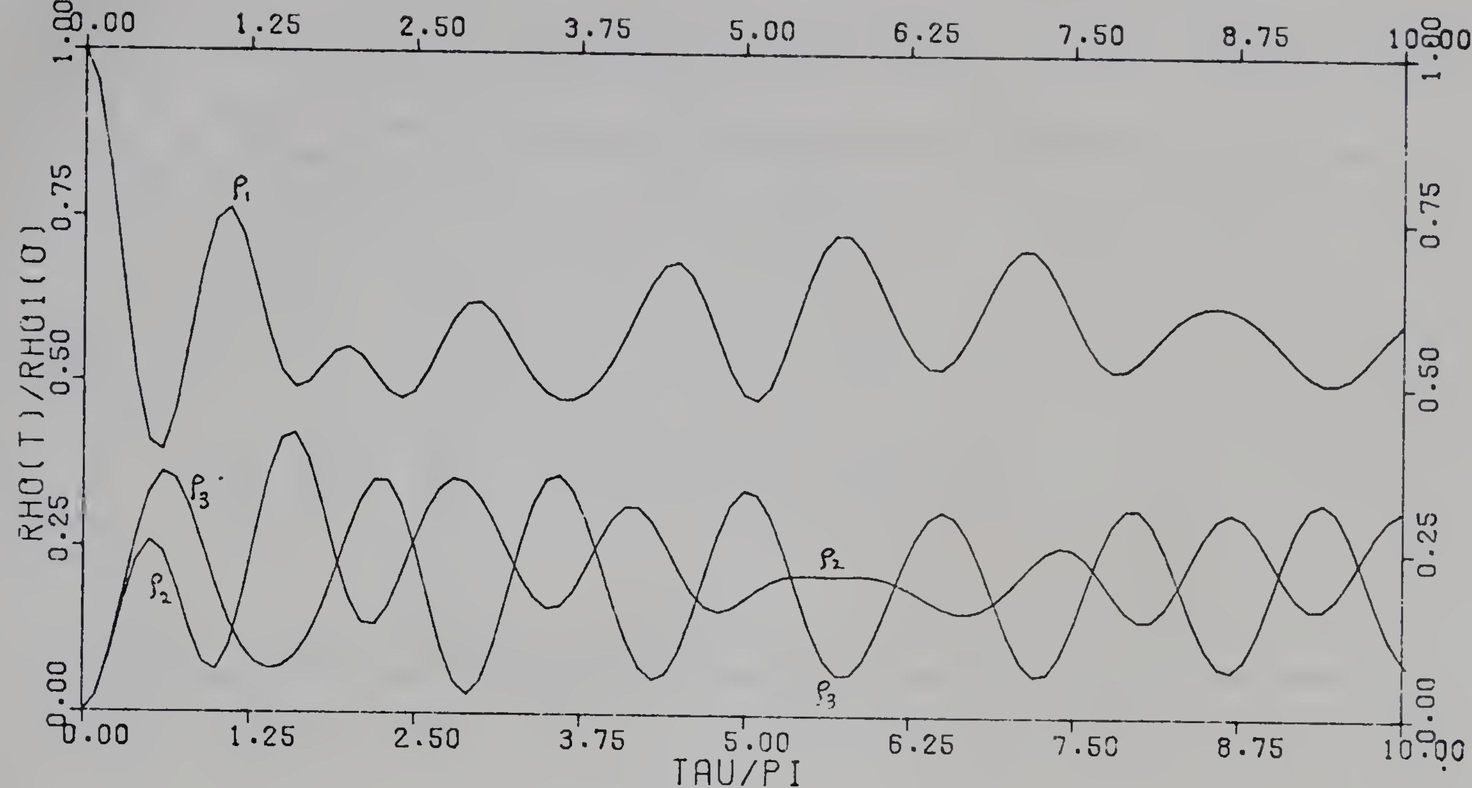


Fig. A.12. $r_2 = 2.$; $r_3 = 3.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = U_2 = \delta = 1.$

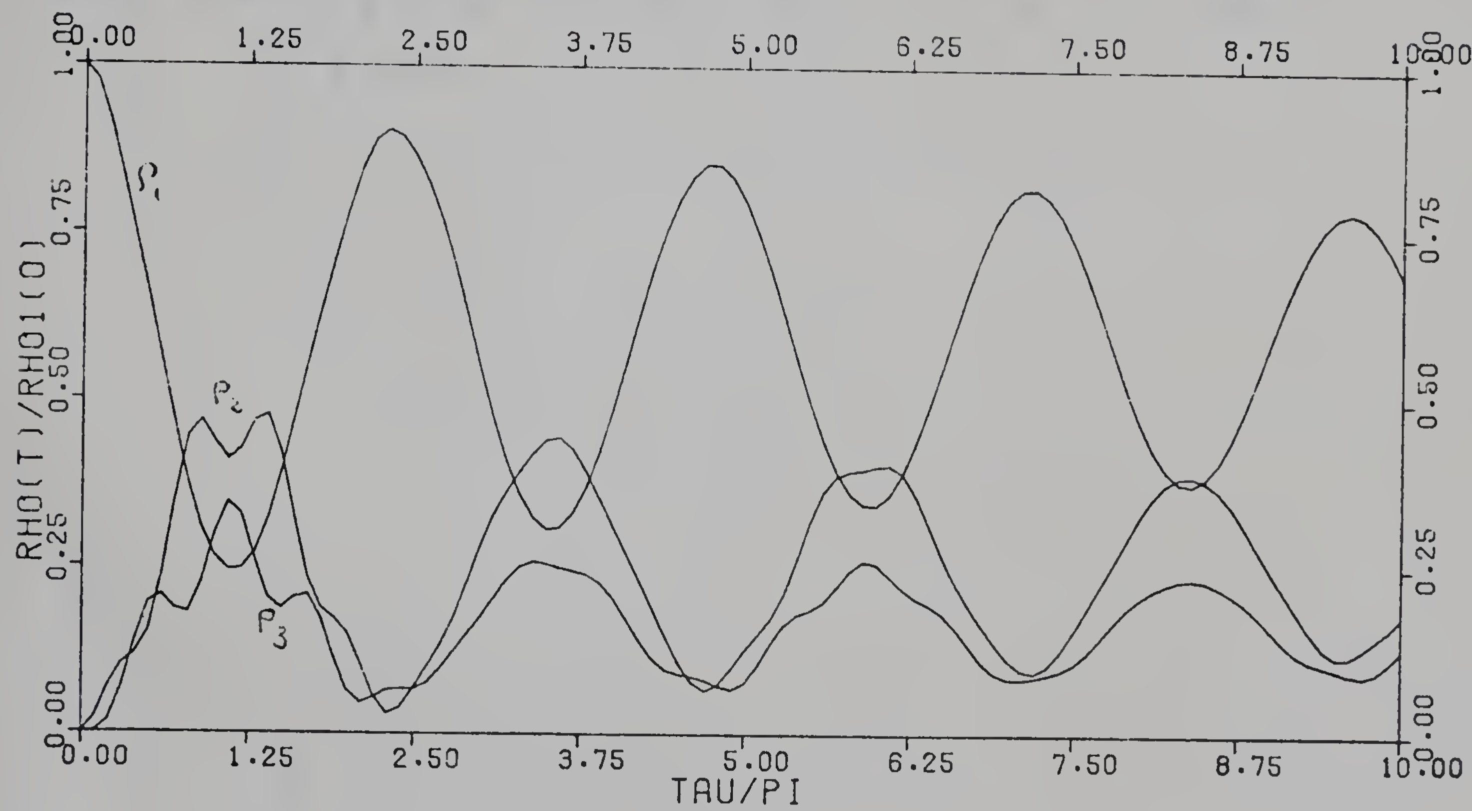
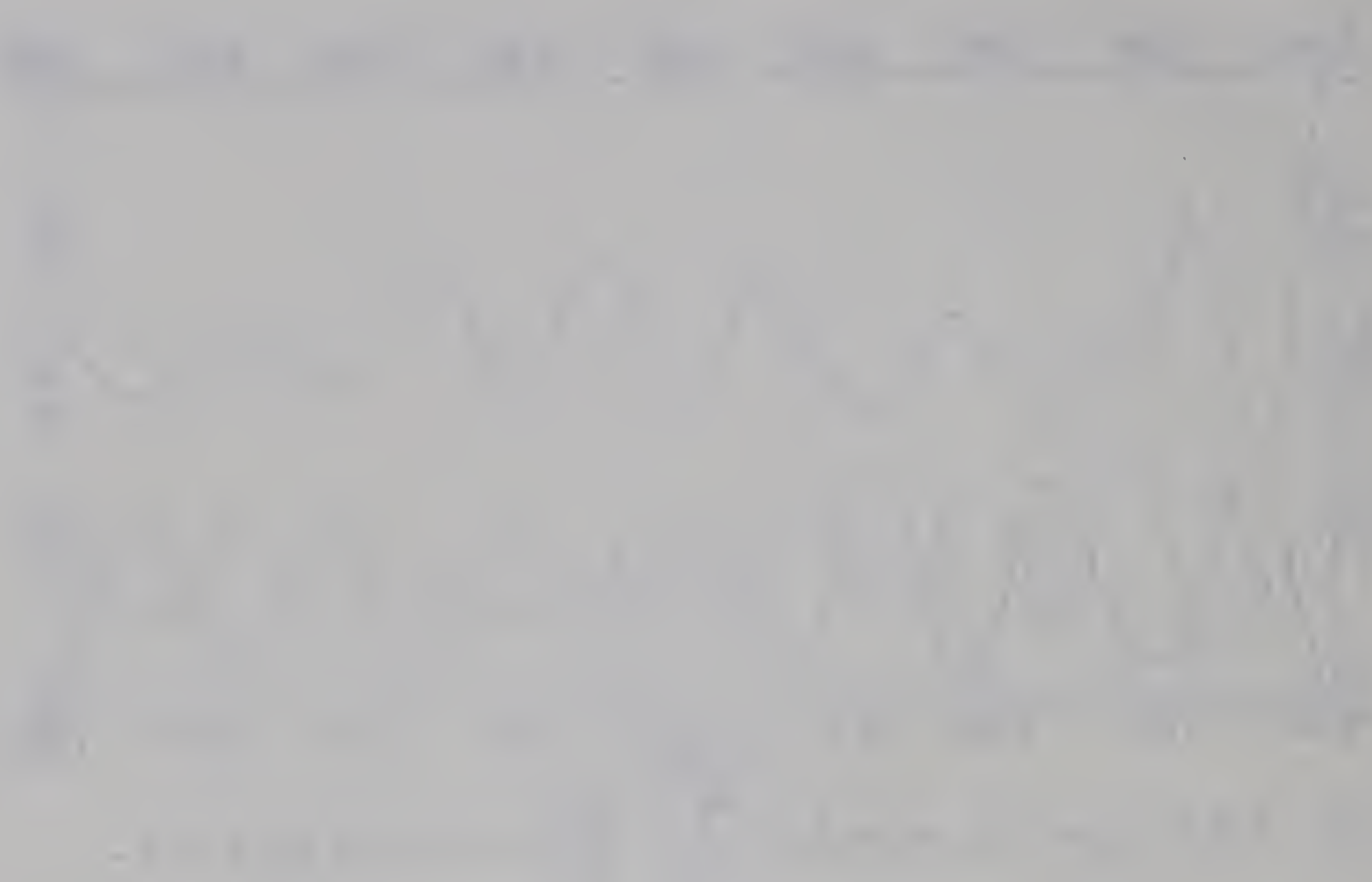
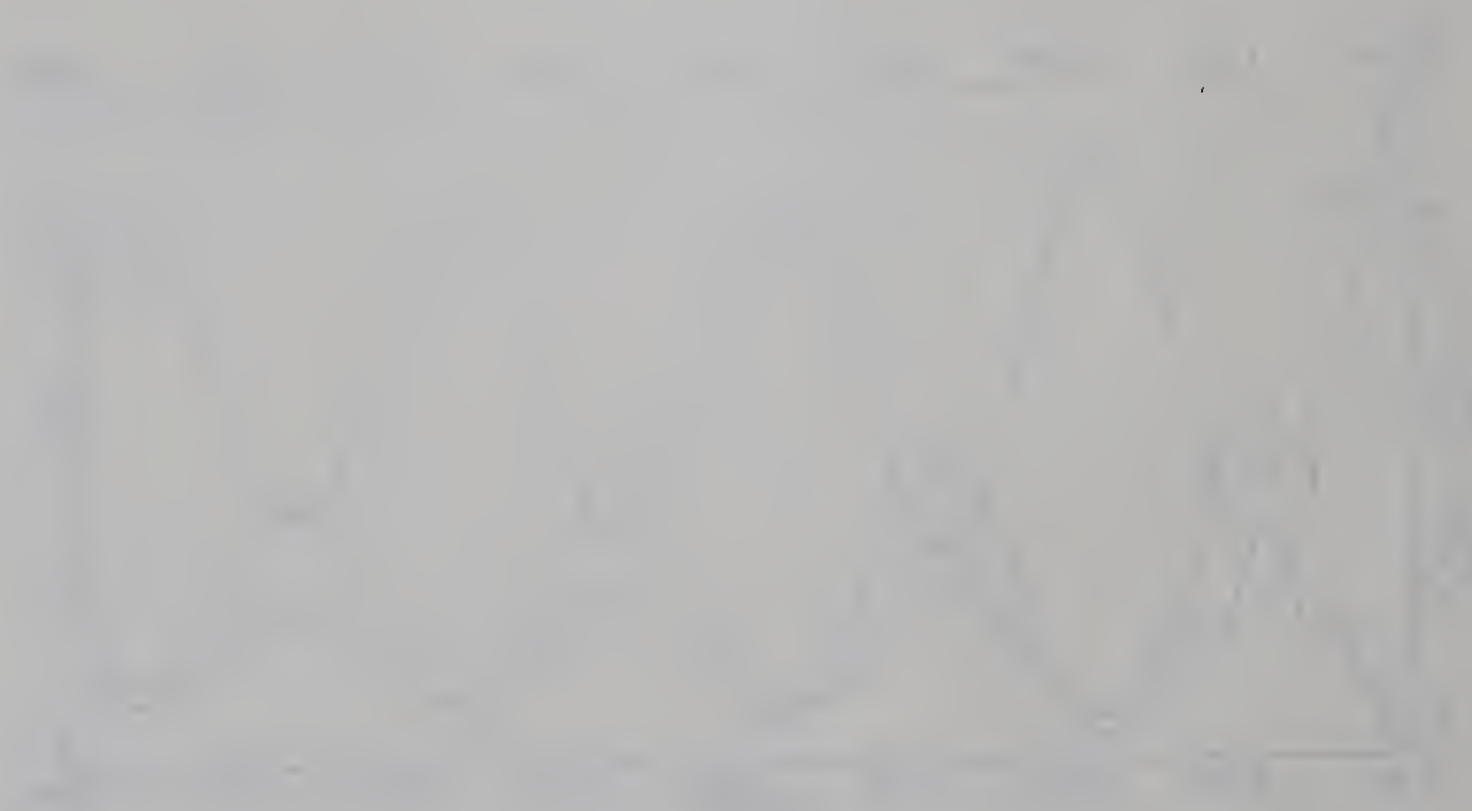


Fig. A.13. $r_2 = 3.$; $r_3 = 5.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 0.$; $U_1 = 0.$; $U_2 = 3.$; $\delta = 1.$

Note the similarity to Fig. 2.6(c). This is because U and P_i, i = 1,2,3 are the same at $x = \delta^{-\frac{1}{2}}$ for both.





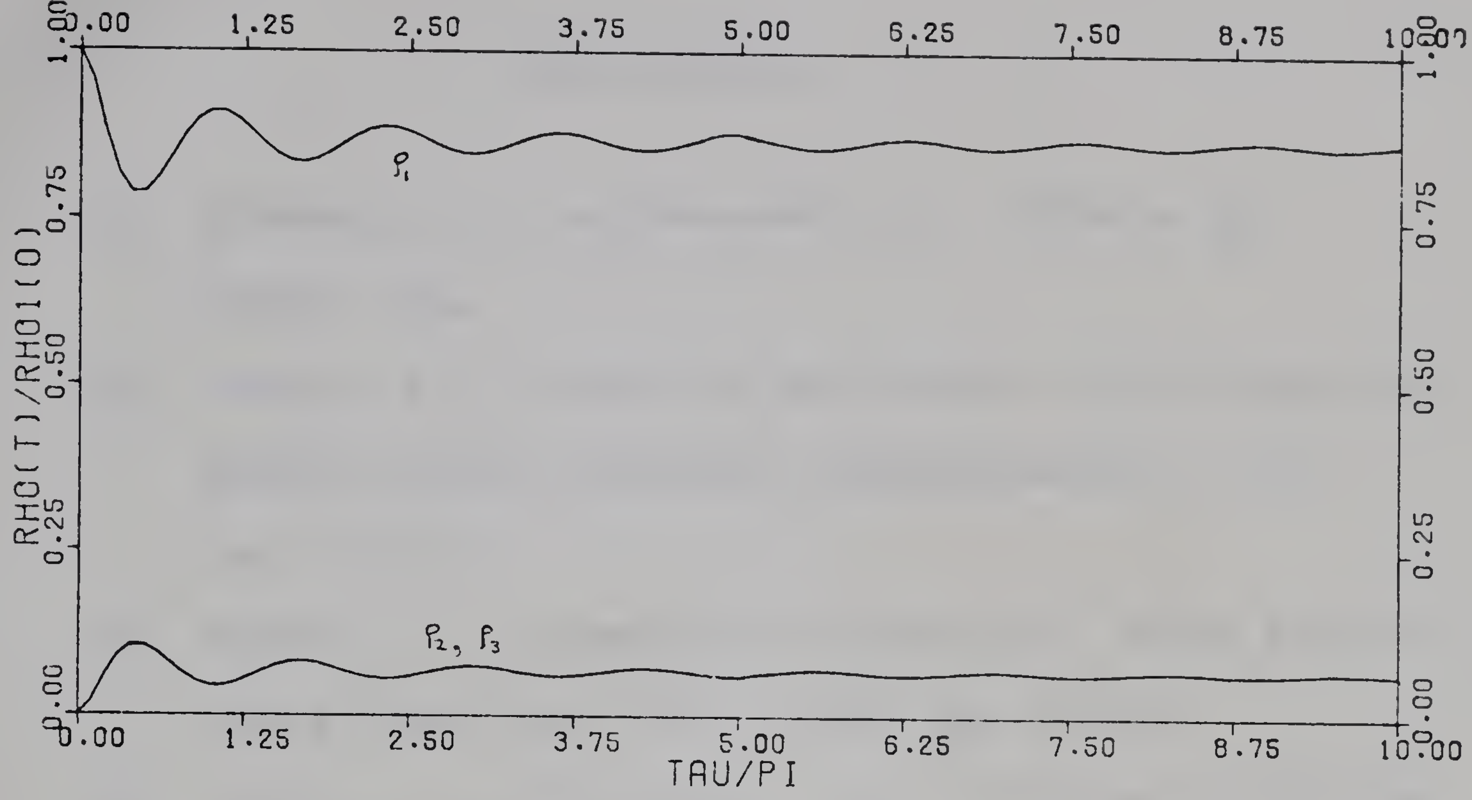


Fig. A.14. $r_2 = r_3 = 1.$; $\frac{m_1}{m_2} = \frac{m_1}{m_3} = 5.$; $U_1 = 1.$; $U_2 = 0.$;
$\delta = 1.$

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